Disjunction Property and Finite Model Property for An Intuitionistic Epistemic Logic

Yoichi Hirai *

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Abstract

We show finite model property, disjunction property and decidability for an intuitionistic epistemic logic **IEC**. Intuitionistic logic is originally a formalisation of a single mathematician whose knowledge increases over time. The logic **IEC** formalises multiple agents that communicate asynchronously and whose knowledge increases over time. Although soundness and strong completeness for **IEC** for a Kripke semantics was straightforward, finite model property and disjunction property required a special technique called modal map on sets of formulas.

1 Introduction

This work extends Hirai [1]. Although the definition of logic **IEC**, soundness and strong completeness are given by Hirai [1], disjunction property and finite model property are new.

Intuitionistic Epistemic Logic Agents in asynchronous systems can obtain knowledge about other agents only by receiving some constructions from them, not by waiting for a fixed length of time. This specific style of knowledge, where obtaining knowledge requires obtaining a physical construction, is the same as the style of knowledge of intuitionistic, constructive reasoners. That is the reason why we deliberately choose intuitionistic not classical meanings for the basic logical connectives, especially implication $\supset$ and disjunction $\vee$. The Kripke model for asynchronous communication can be seen as a description of agents passing around constructions that ensure propositions.

We extend the language of intuitionistic propositional logic with a unary operator K_a, whose meaning can be expressed as: a proof of $K_a\varphi$ is a construction that witnesses agent a's acknowledgement of a proof of φ and also contains the acknowledged proof. This meaning is different from that of classical epistemic logic where the meaning of K_a can be expressed as: $K_a\varphi$ is valid if and only if φ is valid in all possible worlds that agent a thinks possible.

One advantage of our meaning of K_a over that of classical meaning is that it can express communication without the help of another modality. Namely, in our meaning, a proof of K_bK_aP is a construction that is passed from agent a to agent b. On the other hand, in classical meaning, the same formula expresses nothing about communication: K_bK_aP is valid when P is valid in all possible worlds that agent b in any possible world that agent a thinks possible thinks possible.

Intuitionistic logic can be seen as a logic describing an agent whose knowledge increases over time. The logic **IEC** can be seen as a logic describing multiple agents that asynchronously communicate with each other and increase their knowledge. Although **IEC** deals with communication, the logic has only epistemic modalities so that it has simpler syntax than many other logics for communication.

*The Department of Computer Science, The University of Tokyo

1.1 Preliminaries and Notations

We assume inductive definitions using BNF and coinductive definition. $\mathcal{P}(X)$ denotes the powerset of X. For a symbol or a sequence of symbols p, p^+ denotes repetition of p more than zero times and p^* denotes repetition of p more than or equal to zero times.

2 Definitions and Old Results

In this section, we review a logic called intuitionistic epistemic logic proposed by Hirai [1]. The logic has epistemic modality K_a in addition to ordinary logical connectives $(\wedge, \vee, \supset, \perp)$ of propositional logic. We explain the meaning of the new modality K_a informally, by extending the Brouwer–Heyting–Kolmogorov interpretation (BHK-interpretation) [3] of logical connectives.

2.1 Formulas

We fix a countably infinite set of propositional symbols $PVar$ and a finite set of agents A. Let $P, Q, \ldots$ run over the propositional symbols.

Definition 1. *We define a formula φ by the BNF:*

$$\varphi ::= \perp \mid P \mid (K_a\varphi) \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid (\varphi \supset \varphi)$$

where $a \in A$ stands for an agent.

The unary operators connect more strongly than the binary operators. We sometimes omit the parentheses when no confusion occurs. We use $=$ for syntactic equality of formulas. The notation $(\neg\varphi)$ stands for $(\varphi \supset \perp)$. For a sequence of formulas $\Gamma = (\varphi_i)_{i \in I}$ or a set of formulas Γ, the notation $K_a\Gamma$ stands for the sequence $(K_a\varphi_i)_{i \in I}$ or the set $\{K_a\varphi \mid \varphi \in \Gamma\}$ respectively.

2.2 Informal Explanation by BHK-Interpretation

Intuitionistic meanings for logical connectives can be presented as sentences called BHK-interpretation. In this paper, we consider extending BHK-interpretation with another clause for epistemic modality:

(HK) A proof of $K_a\varphi$ is a construction that witnesses agent a's acknowledgement of a proof of φ and also contains the acknowledged proof.

We choose to regard knowledge as acknowledgement of proofs so that the modality K_a informally describes knowledge of agent a. The formalisation of knowledge is different from that in classical epistemic logic, where knowledge is described as a limitation on the ability to distinguish possible worlds.

2.3 Deduction System

We give a proof system of **IEC** in natural deduction. Most of the rules are common with intuitionistic propositional logic while some rules are added to define the meaning of the K_a modality.

Definition 2. *We define the proof system of* **IEC** *by Figure 1.*

Rationales for the rules on modalities While the rules (T), (ispec) and (nec) are admissible in classical epistemic logic, we have an additional rule $(\vee K)$ which needs explanation. In this paragraph, we are going to give a rationale for the rule $(\vee K)$ with the help of BHK-interpretation given in Section 2.2. A proof for the premise of the rule $(\vee K)$ is a construction that witnesses agent a's acknowledgement of a proof of $\varphi \vee \psi$. Since a proof of $\varphi \vee \psi$ is either a proof of φ or a proof of psi, agent a's acknowledge of a proof of $\varphi \vee \psi$ implies either agent a's acknowledgement of a proof of φ or agent a's acknowledgement of a proof of ψ.

Also, we are informally assuming logical omniscience of the agents by rule (nec), that is, we assume agents have complete command on intuitionistic epistemic logic so that they acknowledge every formulas deducible from the set of formulas they acknowledge. We do not try to convince that every conceivable agent has logical omniscience. We only speculate that agents without logical omniscience are hard to represent in a formal system.

Notational conventions For a set of formula Γ and a formula φ, $\Gamma \vdash \varphi$ denotes a relation where there

$$(\text{ax})\ \frac{}{\varphi \vdash \varphi} \qquad (\text{w})\ \frac{\Gamma \vdash \varphi}{\psi,\, \Gamma \vdash \varphi} \qquad (\text{c})\ \frac{\varphi,\, \varphi,\, \Gamma \vdash \varphi'}{\varphi,\, \Gamma \vdash \varphi'} \qquad (\text{e})\ \frac{\Gamma, \varphi, \psi,\, \Gamma' \vdash \theta}{\Gamma,\, \psi, \varphi,\, \Gamma' \vdash \theta}$$

$$(\wedge\text{-E}_0)\ \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi} \qquad (\wedge\text{-I})\ \frac{\Gamma \vdash \varphi \qquad \Gamma' \vdash \psi}{\Gamma, \Gamma' \vdash \varphi \wedge \psi} \qquad (\wedge\text{-E}_1)\ \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi}$$

$$(\vee\text{-I}_0)\ \frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi} \qquad (\vee\text{-E})\ \frac{\Gamma \vdash \psi_0 \vee \psi_1 \qquad \Gamma,\, \psi_0 \vdash \varphi \qquad \Gamma,\, \psi_1 \vdash \varphi}{\Gamma \vdash \varphi} \qquad (\vee\text{-I}_1)\ \frac{\Gamma \vdash \varphi}{\Gamma \vdash \psi \vee \varphi}$$

$$(\supset\text{-I})\ \frac{\varphi,\, \Gamma \vdash \psi}{\Gamma \vdash \varphi \supset \psi} \qquad (\supset\text{-E})\ \frac{\Gamma \vdash \psi_0 \supset \psi_1 \qquad \Gamma \vdash \psi_0}{\Gamma \vdash \psi_1} \qquad (\bot\text{-E})\ \frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi} \qquad (\text{T})\ \frac{}{K_a\varphi \vdash \varphi}$$

$$(\text{ispec})\ \frac{}{K_a\varphi \vdash K_a K_a \varphi} \qquad (\text{nec})\ \frac{\Gamma \vdash \varphi}{K_a\Gamma \vdash K_a\varphi} \qquad (\vee K)\ \frac{}{K_a(\varphi \vee \psi) \vdash K_a\varphi \vee K_a\psi}$$

Figure 1: Deduction rules of **IEC**. (ax) stands for axiom, (w) for weakening, (c) for contraction, (e) for exchange, (ispec) for introspection and (nec) for necessitation. ($\diamondsuit$-I) denotes the introduction rule for connective $\diamondsuit$. ($\diamondsuit$-E) denotes the elimination rule for connective $\diamondsuit$.

is such a finite sequence Γ_0 that $\Gamma_0 \vdash \varphi$ is deducible and that Γ_0 contains only formulas in Γ.

2.4 Semantics

We define validity of a formula on a state in a model. A model is a Kripke model for propositional intuitionistic logic equipped with an additional mapping $f_a : W \to W$ for each agent $a \in A$ where W is the set of possible states.

Definition 3. *A model $\langle W, \preceq, (f_a)_{a\in A}, \rho\rangle$ is a tuple with following properties:*

1. *$\langle W, \preceq\rangle$ is a partial order,*
2. *$f_a\colon W \to W$ is a function satisfying*
   - *(a) (descendance) $f_a(w) \preceq w$,*
   - *(b) (idempotency) $f_a(f_a(w)) = f_a(w)$, and*
   - *(c) (monotonicity) $w \preceq v$ implies $f_a(w) \preceq f_a(v)$*

   for all $v, w \in W$,
3. *$\rho\colon PVar \to \mathcal{P}(W)$ is a function such that each $\rho(P)$ is upward-closed with respect to $\preceq$, i.e., $w' \succeq w \in \rho(P)$ implies $w' \in \rho(P)$.*

Definition 4. *We define the validity relation $\models$ of a model $\langle W, \preceq, (f_a)_{a\in A}, \rho\rangle$, a state $w \in W$ of the model and a formula φ. Let us fix a model $M = \langle W, \preceq, (f_a)_{a\in A}, \rho\rangle$. The definition of $M, w \models \varphi$ is inductive on the structure of φ.*

(Case $\varphi = \bot$) *$M, w \models \bot$ never holds.*

(Case $\varphi = P$) *$M, w \models P$ if and only if $w \in \rho(P)$.*

(Case $\varphi = K_a\psi$) *$M, w \models K_a\psi$ if and only if $M, f_a(w) \models \psi$.*

(Case $\varphi = \psi_0 \wedge \psi_1$) *$M, w \models \psi_0 \wedge \psi_1$ if and only if both $M, w \models \psi_0$ and $M, w \models \psi_1$ hold.*

(Case $\varphi = \psi_0 \vee \psi_1$) *$M, w \models \psi_0 \vee \psi_1$ if and only if either $M, w \models \psi_0$ or $M, w \models \psi_1$ holds.*

(Case $\varphi = \psi_0 \supset \psi_1$) *$M, w \models \psi_0 \supset \psi_1$ if and only if for any $w' \in W$ with $w' \succeq w$, the validity $M, w' \models \psi_0$ implies the validity $M, w' \models \psi_1$.*

Theorem 5 (Kripke monotonicity)**.** *$M, w \models \varphi$ and $w \preceq v$ imply $M, v \models \varphi$.*

Proof. By routine structural induction on φ. Monotonicity of f_a is used. □

Notation 6. *For a model M and a state w of the model, we write $M, w \models \Gamma$ when the validity $M, w \models \varphi$ holds for any formula φ in Γ.*

Notation 7. *$\Gamma \models \varphi$ stands for the relation of formula sequences Γ and a formula φ that holds if and only if for any model M and $w \in M$, $M, w \models \Gamma$ implies $M, w \models \varphi$.*

Definition 8. *$\Gamma \models \varphi$ stands for the relation of a set of a formulas Γ and a formula φ where $M, w \models \Gamma$ implies $M, w \models \varphi$ for any model M and a state $w \in M$.*

For a sequence of formulas Γ, we let $u(\Gamma)$ denote the set of formulas appearing in Γ. We abbreviate $u(\Gamma) \models \varphi$ into $\Gamma \models \varphi$. We will sometimes write Γ instead of $u(\Gamma)$ for the sake of brevity.

Theorem 9 (Soundness and strong completeness)**.** $\Gamma \vdash \varphi \Longleftrightarrow \Gamma \models \varphi$.

Proof. Standard. The proof is in Hirai [1]. □

3 Disjunction Property

We modify Aczel's slash and prove disjunction property. We referred Troelstra and van Dalen's textbook [3, 3.5] for the proof of disjunction property of intuitionistic propositional logic.

The main originality of this paper is the following definition of the function f_a we call the modal map. Informally, for a set Γ of formulas, $f_a(\Gamma)$ is agent a's view of the set Γ.

Definition 10. *For an agent $a \in A$, we define two functions $g_a, f_a \colon \mathcal{P}(\mathbf{Fml}) \to \mathcal{P}(\mathbf{Fml})$ as*

$$\begin{aligned} g_a(\Gamma) &= \{\varphi \in \mathbf{Fml} \mid (K_a)^+\varphi \in \Gamma \textit{ and } \varphi \\ &\qquad \textit{does not begin with } K_a\}, \\ f_a(\Gamma) &= g_a(\Gamma) \cup K_a g_a(\Gamma) \cup \{\varphi \in \mathbf{Fml} \mid \Gamma \vdash \bot\}. \end{aligned}$$

Proposition 11. $\Gamma \subseteq \Delta$ *implies* $g_a(\Gamma) \subseteq g_a(\Delta)$.

Proof. By the form of definition of g_a in Definition 10. □

Proposition 12. $g_a(\Delta \cup \Gamma) = g_a(\Delta) \cup g_a(\Gamma)$.

Proof. By the form of definition of g_a in Definition 10. □

Proposition 13. $f_a(\Delta \cup \Gamma)$ *is equal to* $f_a(\Delta) \cup f_a(\Gamma)$ *provided* $\Delta \cup \Gamma \nvdash \bot$.

Proof.

$$\begin{aligned} f_a(\Delta \cup \Gamma) &= g_a(\Delta \cup \Gamma) \cup K_a g_a(\Delta \cup \Gamma) \\ &= g_a(\Delta) \cup g_a(\Gamma) \cup K_a g_a(\Delta) \cup K_a g_a(\Gamma) \\ &= f_a(\Delta) \cup f_a(\Gamma). \end{aligned}$$

□

Proposition 14. $\Gamma \subseteq \Delta$ *implies* $f_a(\Gamma) \subseteq f_a(\Delta)$.

Proof. If $\Delta \vdash \bot$, $f_a(\Gamma) \subseteq \mathbf{Fml} = f_a(\Delta)$. Otherwise, there exists a set Γ' with $\Gamma \cup \Gamma' = \Delta$. Using Proposition 13 suffices. □

Proposition 15. *For any* $\varphi \in K_a f_a(\Gamma)$, $\Gamma \vdash \varphi$ *holds.*

Proof. $\varphi = K_a\psi$ where $\psi \in f_a(\Gamma) = g_a(\Gamma) \cup K_a g_a(\Gamma) \cup \{\varphi \in \mathbf{Fml} \mid \Gamma \vdash \bot\}$.

(Case $\psi \in g_a(\Gamma)$**)** By definition of g_a, $(K_a)^+\psi \in \Gamma$. By rule (T), $\Gamma \vdash K_a\psi$. This is what we sought: $\Gamma \vdash \varphi$.

(Case $\psi \in K_a g_a(\Gamma)$**)** $\psi = K_a\psi'$ where $\psi' \in g_a(\Gamma)$. By the same argument, $\Gamma \vdash K_a\psi'$. By rule (inspec), $\Gamma \vdash K_a K_a\psi'$. This is what we sought: $\Gamma \vdash \varphi$.

(Case $\Gamma \vdash \bot$**)** By rule ($\bot$-E), $\Gamma \vdash \varphi$ holds.

□

Proposition 16. *For any* $\varphi \in f_a(\Gamma)$, $\Gamma \vdash \varphi$ *holds.*

Proof. $K_a\varphi \in K_a f_a(\Gamma)$. By Proposition 15, $\Gamma \vdash K_a\varphi$ holds. By rule (T), deducibility $\Gamma \vdash \varphi$ holds. □

Proposition 17. $f_a(f_a(\Gamma)) = f_a(\Gamma)$.

Proof. If $\Gamma \nvdash \bot$, by Proposition 16, $f_a(\Gamma) \nvdash \bot$ also holds.

$$\begin{aligned} f_a(f_a(\Gamma)) &= f_a(g_a(\Gamma) \cup K_a g_a(\Gamma)) && \text{(def. of } f_a) \\ &= f_a(g_a(\Gamma)) \cup f_a(K_a g_a(\Gamma)) && \text{(Prop. 13)} \\ &= \emptyset \cup f_a(\Gamma) && \text{(def. of } f_a, g_a) \\ &= f_a(\Gamma). \end{aligned}$$

Otherwise, if $\Gamma \vdash \bot$, $f_a(\Gamma) = \mathbf{Fml} = f_a(f_a(\Gamma))$. □

Definition 18. *We define the slash relation* $|$ *as follows:*

$$\begin{aligned}
\Gamma \mid \bot &\iff \Gamma \vdash \bot, \\
\Gamma \mid P &\iff \Gamma \vdash P, \\
\Gamma \mid K_a\varphi &\iff f_a(\Gamma) \mid \varphi \\
\Gamma \mid \varphi \wedge \psi &\iff \Gamma \mid \varphi \text{ and } \Gamma \mid \psi, \\
\Gamma \mid \varphi \vee \psi &\iff \Gamma \mid \varphi \text{ or } \Gamma \mid \psi, \\
\Gamma \mid \varphi \supset \psi &\iff \Delta \mid \varphi \text{ implies } \Delta \mid \psi \text{ for any } \Delta \supseteq \Gamma \\
&\qquad \text{and also } \Gamma \vdash \varphi \supset \psi.
\end{aligned}$$

Lemma 19. $\Gamma \mid \varphi \Rightarrow \Gamma \vdash \varphi$.

Proof. By induction on φ.

(Case $\varphi = \bot$**)(Case** $\varphi = P$**)** By definition of $|$.

(Case $\varphi = K_a\psi$**)** The assumption $\Gamma \mid K_a\psi$ is equivalent to $f_a(\Gamma) \mid \psi$. By induction hypothesis, $f_a(\Gamma) \vdash \psi$. By rule (nec), $K_a f_a(\Gamma) \vdash K_a\psi$. By Proposition 15, the deducibility $\Gamma \vdash K_a\psi$ holds.

(Case $\varphi = \psi_0 \wedge \psi_1$**)** $\Gamma \mid \psi_0 \wedge \psi_1$. By definition of $|$, Both $\Gamma \mid \psi_0$ and $\Gamma \mid \psi_1$ hold. By induction hypothesis, both $\Gamma \vdash K_x\psi_0$ and $\Gamma \vdash K_x\psi_1$ hold. By logic, $\Gamma \vdash K_x(\psi_0 \wedge \psi_1)$ holds.

(Case $\varphi = \psi_0 \vee \psi_1$**)** Similar to the case above.

(Case $\varphi = \psi_0 \supset \psi_1$**)** By definition of $|$.

□

Lemma 20. $\Gamma \mid \varphi$ *and* $\Gamma \subseteq \Delta$ *imply* $\Delta \mid \varphi$.

Proof. By induction on φ.

(Case $\varphi = \bot$**) (Case** $\varphi = P$**) (Case** $\varphi = \psi_0 \supset \psi_1$**)** By definition of the slash relation $|$.

(Case $\varphi = \psi_0 \wedge \psi_1$**) (Case** $\varphi = \psi_0 \vee \psi_1$**)** Directly from induction hypotheses.

(Case $\varphi = K_a\psi$**)** By Proposition 14, $f_a(\Gamma) \subseteq f_a(\Delta)$ holds. By induction hypothesis, $f_a(\Gamma) \mid \psi$ implies $f_a(\Delta) \mid \psi$, which is equivalent to $\Delta \mid \varphi$ holds.

□

Lemma 21. *For any set* Γ *of formulas with* $\Gamma \mid \psi$ *for all* $\psi \in \Gamma$, $\varphi \in g_a(\Gamma)$ *implies* $f_a(\Gamma) \mid \varphi$.

Proof. By definition of g_a, $(K_a)^{(n)}\varphi \in \Gamma$ for some $n \geq 1$, where $(K_a)^{(n)}$ denotes an n-time repetition of K_a's. By assumption, $\Gamma \mid (K_a)^{(n)}\varphi$. By definition of $|$, $f_a^{(n)}(\Gamma) \mid \varphi$. Since f_a is idempotent (Proposition 17), $f_a(\Gamma) \mid \varphi$. □

Definition 22. *A hereditary* f*-closed set* Γ *is coinductively defined as:* Γ *is a hereditary* f*-closed set if and only if* $f_a(\Gamma)$ *is hereditary* f*-closed and* $f_a(\Gamma) \subseteq \Gamma$ *for all* $a \in A$.

For example, the set $\Gamma = \{K_bK_aK_aP\}$ is not hereditary f-closed because $f_b(\Gamma)$ is not hereditary f-closed. $f_b(\Gamma)$ is not hereditary f-closed because $K_aP \in f_a(f_b(\Gamma))$ while $K_aP \notin f_b(\Gamma)$.

Lemma 23. *For any hereditary* f*-closed set* Γ *and a formula* φ, $\Gamma \vdash \bot$ *implies* $\Gamma \mid \varphi$.

Proof. By induction on φ.

(Case $\varphi = \bot$**) (Case** $\varphi = P$**)** $\Gamma \vdash \varphi$ implies $\Gamma \mid \varphi$ because φ is atomic.

(Case $\varphi = K_a\psi$**)** Since $f_a(\Gamma)$ is also hereditary f-closed and $\bot \operatorname{in} f_a(\Gamma)$, by induction hypothesis, $f_a(\Gamma) \mid \psi$. This is equivalent to $\Gamma \mid K_a\psi$.

(Case $\varphi = \psi_0 \wedge \psi_1$**) (Case** $\varphi = \psi_0 \vee \psi_1$**)** Directly from induction hypothesis.

(Case $\varphi = \psi_0 \supset \psi_1$**)** By rule ($\bot$-E), $\Gamma \vdash \psi_0 \supset \psi_1$. For all $\Delta \supset \Gamma$, by induction hypothesis, $\Delta \mid \psi_1$ holds. These two facts show $\Delta \mid \psi_0 \supset \psi_1$.

□

Lemma 24. *For any hereditary* f*-closed set* Γ *of formulas, if* $\Gamma \mid \psi$ *for all* $\psi \in \Gamma$, $f_a(\Gamma) \mid \varphi$ *for all* $\varphi \in f_a(\Gamma)$.

Proof. By induction on the structure of φ. However, most cases are uniformly treated in the last clause.

(Case $\varphi = K_x\psi$**)** Assume $K_x\psi \in f_a(\Gamma) = g_a(\Gamma) \cup K_a g_a(\Gamma) \cup \{\theta \in \mathbf{Fml} \mid \Gamma \vdash \bot\}$.

(Case $K_x\psi \in g_a(\Gamma)$) By Lemma 21, $f_a(\Gamma) \mid K_x\psi$.

(Case $K_x\psi \in K_a g_a(\Gamma)$) Note $x = a$. By Lemma 21, $f_a(\Gamma) \mid \psi$ holds. Since f_a is idempotent, $f_a(f_a(\Gamma)) \mid \psi$ holds. By definition of $\mid$, $f_a(\Gamma) \mid K_a\psi$.

(Case $\Gamma \vdash \bot$) $f_a(\Gamma) \vdash \bot$ also holds. By Lemma 23, $f_a(\Gamma) \mid \varphi$ holds.

(Other cases) Assume $\varphi \in f_a(\Gamma) = g_a(\Gamma) \cup K_a g_a(\Gamma) \cup \{\theta \in \mathbf{Fml} \mid \Gamma \vdash \bot\}$. If $\Gamma \vdash \bot$, by Lemma 23 and definition of f_a, $f_a(\Gamma) \mid \varphi$. Otherwise, since the formula φ does not begin with K_a, $\varphi \in g_a(\Gamma)$. By Lemma 21, $f_a(\Gamma) \mid \varphi$.

□

Lemma 25. $\Gamma \mid K_a\varphi \Rightarrow \Gamma \mid \varphi$ *if* Γ *is* f_a*-closed.*

Proof. Immediate from Lemma 20. □

Lemma 26. $\Gamma \mid \psi$ *and* $\Gamma \cup \{\psi\} \mid \varphi$ *imply* $\Gamma \mid \varphi$.

Proof. By induction on φ.

(Case $\varphi = \bot$) (Case $\varphi = P$) Since $\Gamma \mid \psi$, by Lemma 19, the deducibility $\Gamma \vdash \psi$ holds. Likewise since $\Gamma \cup \{\psi\} \mid \varphi$, the deducibility $\Gamma \cup \{\psi\} \vdash \varphi$ holds. These combined imply $\Gamma \vdash \varphi$. By definition of the slash relation $\mid$, the relation $\Gamma \mid \varphi$ holds because φ is atomic.

(Case $\varphi = \psi_0 \vee \psi_1$) (Case $\varphi = \psi_0 \wedge \psi_1$) Directly from induction hypotheses.

(Case $\varphi = K_a\theta$) Since $\Gamma \cup \{\psi\} \mid K_a\theta$, by definition of the slash relation $\mid$, $f_a(\Gamma \cup \{\psi\}) \mid \theta$ holds. If $\Gamma \cup \{\psi\} \vdash \bot$, by the assumption, $\Gamma \vdash \bot$. Thus, by Lemma 23, $\Gamma \vdash \varphi$ holds. Otherwise, since $f_a(\Gamma \cup \{\psi\}) = f_a(\Gamma) \cup f_a(\{\psi\})$, we have $f_a(\Gamma) \cup f_a(\{\psi\}) \mid \theta$. If $\psi = K_a\psi'$, $\Gamma \mid \psi$ is equivalent to $f_a(\Gamma) \mid \psi'$. By induction hypothesis, $f_a(\Gamma) \mid \theta$. This is equivalent to $f_a(\Gamma) \mid K_a\theta$. This is what we sought: $f_a(\Gamma) \mid \varphi$. Otherwise, if ψ does not begin with K_a, $f_a(\{\psi\}) = \emptyset$. Thus, $f_a(\Gamma) \mid \theta$. This means $\Gamma \mid K_a\theta$.

(Case $\varphi = \psi_0 \supset \psi_1$) Since $\Gamma \cup \{\psi\} \mid \psi_0 \supset \psi_1$, by Lemma 19, $\Gamma \cup \{\psi\} \vdash \psi_0 \supset \psi_1$ holds. In addition to this, $\Delta \cup \{\psi\} \mid \psi_0$ implies $\Delta \cup \{\psi\} \mid \psi_1$ for any $\Delta \supseteq \Gamma$. We claim that $\Delta \mid \psi_0$ implies $\Delta \mid \psi_1$ for any $\Delta \supseteq \Gamma$. To show that, we assume $\Delta \mid \psi_0$. By Lemma 20, $\Delta \cup \{\psi\} \mid \psi_0$ holds. By assumption, $\Delta \cup \{\psi\} \mid \psi_1$ holds. By induction hypothesis, $\Delta \mid \psi_1$ holds. We have shown that $\Delta \mid \psi_0$ implies $\Delta \mid \psi_1$. In addition to this, by $\Gamma \vdash \psi$ and $\Gamma \cup \{\psi\} \vdash \psi_0 \supset \psi_1$, the deducibility $\Gamma \vdash \psi_0 \supset \psi_1$ holds. The slash relation $\Gamma \mid \psi_0 \supset \psi_1$ has been proved.

□

Lemma 27. *If* Γ *and* Δ *are provably equivalent and satisfy* $f_a(\Gamma) = f_a(\Delta)$ *for all* $a \in A$, $\Gamma \mid \varphi$ *is equivalent to* $\Delta \mid \varphi$.

Proof. By the form of the definition of the slash relation $\mid$. □

Theorem 28. *For any hereditary* f*-closed set* Γ *of formulas, if* $\Gamma \mid \varphi$ *holds for any* $\varphi \in \Gamma$, $\Gamma \vdash \varphi$ *implies* $\Gamma \mid \varphi$.

Proof. By induction on definition of $\Gamma \vdash \varphi$.

(ax) (w) (c) (e) Trivial.

($\wedge$-E$_i$) ($\wedge$-I) ($\vee$-I$_i$) By definition of the slash relation $\mid$.

($\vee$-E)
$$\frac{\Gamma \vdash \psi_0 \vee \psi_1 \qquad \Gamma, \psi_0 \vdash \varphi \qquad \Gamma, \psi_1 \vdash \varphi}{\Gamma \vdash \varphi}$$

By an induction hypothesis, $\Gamma \mid \psi_0 \vee \psi_1$ holds. By definition of the slash relation, either $\Gamma \mid \psi_0$ or $\Gamma \mid \psi_1$ holds.

(Case $\Gamma \mid \psi_0$) By another induction hypothesis, $\Gamma \cup \{\psi_0\} \mid \varphi$ holds. By Lemma 26, $\Gamma \mid \varphi$ holds.

(Case $\Gamma \mid \psi_1$) Similar.

($\supset$-I)
$$\frac{\varphi, \Gamma \vdash \psi}{\Gamma \vdash \varphi \supset \psi}$$

By induction hypothesis, $\varphi \cup \Gamma | \psi$ holds. Thus for any $\Delta \supseteq \Gamma$, $\varphi \cup \Delta \mid \psi$ holds. $\Delta \mid \varphi$ implies $\Delta \mid \psi$ by Lemma 26. This fact and the deducibility $\Gamma \vdash \varphi \supset \psi$ imply $\Gamma \mid \varphi \supset \psi$.

(⊃-E) $\dfrac{\Gamma \vdash \psi_0 \supset \psi_1 \qquad \Gamma \vdash \psi_0}{\Gamma \vdash \psi_1}$

By induction hypothesis, $\Gamma \mid \psi_0 \supset \psi_1$ holds. By definition of the slash relation, $\Gamma \mid \psi_0$ implies $\Gamma \mid \psi_1$. Actually, $\Gamma \mid \psi_0$ holds by an induction hypothesis. Thus, $\Gamma \mid \psi_1$ holds.

(⊥-E) By Lemma 23.

(T) $\dfrac{}{K_a\varphi \vdash \varphi}$
Assume $K_a\varphi \in \Gamma$. By assumption of the theorem, $\Gamma | K_a\varphi$. Since Γ is f_a-closed, by Lemma 25, $\Gamma \mid \varphi$.

(nec) $\dfrac{\Delta \vdash \varphi}{K_a\Delta \vdash K_a\varphi}$

We can assume $K_a\Delta \subseteq \Gamma$ and that $\varphi \in \Gamma$ implies $\Gamma | \varphi$. Also, by induction hypothesis, any Γ' with $\Delta \subseteq \Gamma'$ and $\psi \in \Gamma' \Rightarrow \Gamma' | \psi$, $\Gamma' | \varphi$ holds. Since Δ is a finite sequence, there exists a natural number n with $\Delta \subseteq f_a(\Gamma) \cup K_a f_a(\Gamma) \cup \cdots \cup (K_a)^{(n)} f_a(\Gamma)$. By induction hypothesis, $f_a(\Gamma) \cup K_a f_a(\Gamma) \cup \cdots \cup (K_a)^{(n)} f_a(\Gamma) \mid \varphi$ holds. By Lemma 27, this is equivalent to $f_a(\Gamma) | \varphi$. By definition of $|$, $\Gamma | K_a\varphi$ holds.

$(\vee K)$ $\dfrac{}{K_a(\varphi \vee \psi) \vdash (K_a\varphi) \vee K_a\psi}$

$$\begin{aligned} \Gamma \mid K_a(\varphi \vee \psi) &\iff f_a(\Gamma) \mid \varphi \vee \psi \\ &\iff f_a(\Gamma) \mid \varphi \text{ or } f_a(\Gamma) \mid \psi \\ &\iff \Gamma \mid K_a\varphi \text{ or } \Gamma \mid K_a\psi \\ &\iff \Gamma \mid K_a\varphi \vee K_a\psi \end{aligned}$$

□

Using the apparatus prepared above, we can finally show disjunction property, which is the standard for constructive logic.

Theorem 29 (Disjunction property). *If $\vdash \varphi \vee \psi$ holds, either $\vdash \varphi$ or $\vdash \psi$ holds.*

Proof. Taking $\Gamma = \emptyset$ in Theorem 28, $\vdash \varphi \vee \psi$ implies $\emptyset \mid \varphi$ or $\emptyset \mid \psi$. By Lemma 19, either $\vdash \varphi$ or $\vdash \psi$ holds. □

4 Finite Model Property

This model construction below inspired by Sato's paper [2] and Troelstra and van Dalen's textbook [3]. However, the notion of f'-subformula-closed sets is new and original.

Definition 30. *We modify f_a introduced in the last section (Definition 10) and define f'_a as:*

$$f'_a(\Gamma) = g_a(\Gamma) \cup K_a g_a(\Gamma).$$

Definition 31. *For a set of formulas Ω, a set of formulas $\Gamma \subseteq \Omega$ is Ω-saturated if and only if*

1. *Γ is Ω-deductively closed, i.e., $\Gamma \vdash \varphi \in \Omega \Rightarrow \varphi \in \Gamma$,*
2. *$\Gamma \vdash \varphi \vee \psi \Rightarrow \Gamma \vdash \varphi$ or $\Gamma \vdash \psi$ if $\varphi, \psi \in \Omega$,*
3. *$\Gamma \nvdash \bot$.*

Definition 32. *A hereditary f'-subformula-closed set Γ is coinductively defined as: Γ is a hereditary f'-subformula-closed set if and only if $f'_a(\Gamma)$ is hereditary f'-closed, Γ is closed for taking subformulas and $f'_a(\Gamma) \subseteq \Gamma$.*

Definition 33. *We define $s_a(\varphi)$ inductively on φ:*

$$s_a(\varphi) = \begin{cases} s_a(K_a\psi) & (\textit{if } \varphi = K_aK_a\psi), \\ \varphi & (\textit{otherwise}). \end{cases}$$

The function s_a replaces every K_aK_a with K_a repeatedly so that there are no K_aK_a occurrences left.

Lemma 34. *For a hereditary f'-subformula-closed set Ω, if Γ is an Ω-saturated set, $f_a(\Gamma)$ is an $f'_a(\Omega)$-saturated set.*

Proof. We first make sure that $f_a(\Gamma)$ is a subset of $f'_a(\Omega)$. By definition of f_a, $f_a(\Gamma) = g_a(\Gamma) \cup K_a g_a(\Gamma) \cup \{\bot \in \mathbf{Fml} \mid \Gamma \vdash \bot\}$. Since Γ is an Ω-saturated set, $\Gamma \nvdash \bot$ so that $f_a(\Gamma) = g_a(\Gamma) \cup K_a g_a(\Gamma)$. On the other hand, $f'_a(\Omega) = g_a(\Omega) \cup K_a g_a(\Omega)$. Since $g_a(\Gamma) \subseteq g_a(\Omega)$ by Proposition 11, $f_a(\Gamma) \subseteq f'_a(\Omega)$ holds.

We check each condition of Definition 31 to make sure that $f_a(\Gamma)$ is actually an $f'_a(\Omega)$-saturated set.

1. Assume $f_a(\Gamma) \vdash \varphi$ and $\varphi \in f'_a(\Omega)$. $\varphi \in g_a(\Omega) \cup K_a g_a(\Omega)$ holds.

    (Case $\varphi \in g_a(\Omega)$**)** Note that φ does not begin with K_a. By definition of g_a, $(K_a)^+\varphi \in \Omega$. Since Ω is subformula-closed, $K_a\varphi \in \Omega$ holds. By $\Gamma \vdash K_a\varphi$, since Γ is Ω-saturated, $K_a\varphi \in \Gamma$. Thus, $\varphi \in f_a(\Gamma)$.

    (Case $\varphi \in K_a g_a(\Omega)$**)** $\varphi = K_a\varphi'$ and $\varphi' \in g_a(\Omega)$ hold. Note that φ' does not begin with K_a. By definition of g_a, $(K_a)^+\varphi' \in \Omega$. This implies $K_a\varphi' \in \Omega$. Since $\Gamma \vdash K_a K_a \varphi'$, $\Gamma \vdash K_a\varphi'$ holds. Thus, since Γ is Ω-saturated, $K_a\varphi' \in \Gamma$ holds. This means $\varphi = K_a\varphi' \in f_a(\Gamma)$.

2. Assume $f_a(\Gamma) \vdash \varphi \vee \psi$ and $\varphi, \psi \in f'_a(\Omega)$. By rule (nec), $K_a f_a(\Gamma) \vdash K_a(\varphi \vee \psi)$ holds. By Proposition 15, the formulas in $K_a f_a(\Gamma)$ are deducible from Γ. Thus, $\Gamma \vdash K_a(\varphi \vee \psi)$ holds. By rule $(\vee K)$ and the fact that Γ is saturated, either $K_a s_a(\varphi) \in \Gamma$ or $K_a s_a(\psi) \in \Gamma$ holds. We can assume $K_a s_a(\varphi) \in \Gamma$ without loss of generality. This implies $f_a(\Gamma) \vdash \varphi$ and then $\varphi \in f_a(\Gamma)$.

3. Seeking contradiction, assume $f_a(\Gamma) \vdash \bot$. Since $\Gamma \vdash K_a\bot$, the deducibility $\Gamma \vdash \bot$ holds, which contradicts the fact that Γ is an Ω-saturated set.

$\square$

Lemma 35 (Saturation lemma)**.** *For sets of formulas Γ and Ω with $\Gamma \nvdash \varphi$, $\Gamma \subseteq \Omega$ and $\varphi \in \Omega$, there exists an Ω-saturated set Γ^ω with $\Gamma^\omega \nvdash \varphi$ and $\Gamma \subseteq \Gamma^\omega$.*

Proof. Since both $PVar$ and A are countable, we can enumerate all formulas of Ω in a sequence $(\varphi_i)_{i \in \mathbb{N}^+}$. We define Γ^i inductively:

(Case $i = 0$**)** $\Gamma^0 = \Gamma$,

(Case $i > 0$**)**

$$\Gamma^i = \begin{cases} \{\varphi_i\} \cup \Gamma^{i-1} & (\text{if } \{\varphi_i\} \cup \Gamma^{i-1} \nvdash \varphi), \\ \Gamma^{i-1} \cup \{\varphi_i \supset \varphi\} & (\text{o.w. if } \varphi_i \supset \varphi \in \Omega), \\ \Gamma^{i-1} & (\text{o.w.}). \end{cases}$$

Using these Γ^i, we define $\Gamma^\omega = \bigcup_{i \in \omega} \Gamma^i$.

Claim: $\Gamma^\omega \nvdash \varphi$. Seeking contradiction, assume a deducibility $\Gamma^\omega \vdash \varphi$. Since only finite number of formulas in Γ are used to prove φ, there exists a minimal i with $\Gamma^i \vdash \varphi$. Since $\Gamma \nvdash \varphi$, i is not 0. Since $\Gamma^i \neq \Gamma^{i-1}$, either $\Gamma^i = \{\varphi_i\} \cup \Gamma^{i-1}$ or $\Gamma^i = \{\varphi_i \supset \varphi\} \cup \Gamma^{i-1}$ holds. The first case is explicitly forbidden. In the second case, $\Gamma^{i-1}, \varphi_i \supset \varphi \vdash \varphi$ holds. That means $\Gamma^{i-1} \vdash (\varphi_i \supset \varphi) \supset \varphi$. Also, since we could not take the first case, $\Gamma^{i-1}, \varphi_i \vdash \varphi$ holds. That means $\Gamma^{i-1} \vdash \varphi_i \supset \varphi$. By these combined, $\Gamma^{i-1} \vdash \varphi$ holds, which contradicts to the minimality of i. The claim is now proved. $\square$

Claim: Γ^ω is an Ω-saturated set.

Proof of Claim. We check each condition listed in Definition 31:

1. Assume $\Gamma^\omega \vdash \psi \in \Omega$. There is $i \in \mathbb{N}^+$ with $\varphi_i = \psi$. We know that $\Gamma^{i-1} \cup \{\varphi_i\} \nvdash \varphi$. It means $\psi \in \Gamma^\omega$.

2. Assume $\psi_0 \vee \psi_1 \in \Gamma^\omega$ and $\psi_0, \psi_1 \in \Omega$. Seeking contradiction, assume $\psi_0 \notin \Gamma^\omega$ and $\psi_1 \notin \Gamma^\omega$. By construction, both $\Gamma^\omega \vdash \psi_0 \supset \varphi$ and $\Gamma^\omega \vdash \psi_1 \supset \varphi$ hold. Since Γ^ω is deductively closed, by ($\vee$-E) rule, we have $\Gamma^\omega \vdash \varphi$, which contradicts to the previous fact.

3. Since $\Gamma^\omega \nvdash \varphi$, by rule ($\bot$-E), $\Gamma^\omega \nvdash \bot$.

Since $\Gamma^0 = \Gamma$, Γ^ω contains Γ. The lemma is now proved. $\square$

Definition 36 (Canonical model candidate)**.** *For a set of formulas Ω, we define $M^C(\Omega)$ as a tuple $\langle W^C, \preceq^C, (f_a^C)_{a \in A}, \rho^C \rangle$ where:*

- *W^C is the set of pairs of the form (Ω', Γ) where Γ is an Ω'-saturated set and Ω' is a hereditary f'-subformula-closed subset of Ω.*

- $(\Omega', \Gamma) \preceq^{c} (\Omega'', \Delta)$ *if and only if* $\Omega' \subseteq \Omega''$ *and* $\Gamma \subseteq \Delta$,
- $f_a^{c}((\Omega', \Gamma)) = (f'_a(\Omega), f_a(\Gamma))$
- $\rho^{c}(P) = \{(\Omega', \Gamma) \in W^{c} \mid P \in \Gamma\}$.

Lemma 37 (Canonical model)**.** *The tuple M^c is a model.*

Proof. First of all, f_a^{c} is actually a function $W^{\mathrm{c}} \to W^{\mathrm{c}}$ by Lemma 34. We check each condition in Definition 3 to make sure the tuple is actually a model:

1. $\preceq^{\mathrm{c}}$ is a partial order because set theoretic inclusion $\subseteq$ is a partial order.
2. (a) $f_a^{\mathrm{c}}((\Omega', \Gamma)) = (f'_a(\Omega'), f_a(\Gamma))$. Since Ω' is hereditary f'-subset-closed, $f'_a(\Omega') \subseteq \Omega'$ holds. Now, showing $\Gamma \subseteq f_a(\Gamma)$ is enough. Take an arbitrary $\varphi \in f_a(\Gamma)$. Since $\Gamma \not\vdash \bot$, either $\varphi \in g_a(\Gamma)$ or $\varphi \in K_a g_a(\Gamma)$ holds. In either case, $(K_a)^* \varphi \in \Gamma$ holds. That means $\Gamma \vdash \varphi$. Since $\varphi \in \Omega'$, $\varphi \in \Gamma$ holds. Thus we have shown $\Gamma \subseteq f_a(\Gamma)$. This completes the proof of $f_a^{\mathrm{c}}((\Omega', \Gamma)) \preceq^{\mathrm{c}} (\Omega', \Gamma)$.

   (b) By Lemma 17, $f_a(f_a(\Gamma)) = f_a(\Gamma)$ holds. Similar argument gives $f'_a(f'_a(\Omega')) = f'_a(\Omega')$. These combined imply that f_a^{c} is idempotent.

   (c) Both f_a and f'_a are monotonic with respect to set theoretic inclusion. This implies that f_a^{c} is monotonic with respect to $\preceq^{\mathrm{c}}$.
3. Immediate.

□

Lemma 38. *For a state $(\Omega', \Gamma) \in W^c$ in the canonical model M^c and $\varphi \in \Omega'$, φ is an element of Γ if and only if M^c,$(\Omega', \Gamma) \models \varphi$ holds.*

Proof. By induction on φ.

(Case $\varphi = \bot$) Neither side ever holds because Γ is Ω'-saturated.

(Case $\varphi = P$) By definition of ρ^{c} and $\models$, the equivalency $\varphi \in \Gamma \Leftrightarrow (\Omega', \Gamma) \in \rho(P) \Leftrightarrow M^{\mathrm{c}}, (\Omega, \Gamma) \models P$ holds.

(Case $\varphi = \psi_0 \land \psi_1$)(Case $\varphi = \psi_0 \lor \psi_1$) Directly from the induction hypothesis.

(Case $\varphi = K_a \psi$) ($\Rightarrow$) Assume $M^{\mathrm{c}}(\Omega), (\Omega', \Gamma) \models K_a \psi$. By definition of $\models$ and induction hypothesis, $\psi \in f_a(\Gamma) = g_a(\Gamma) \cup K_a g_a(\Gamma)$. If $\psi \in g_a(\Gamma)$, $(K_a)^+ \psi \in \Gamma$ holds. This means $\Gamma \vdash K_a \psi$. Otherwise, if $\psi \in K_a g_a(\Gamma)$, $\psi = K_a \psi'$ where $\psi' \in g_a(\Gamma)$. This means $\Gamma \vdash K_a \psi'$ and consequently $\Gamma \vdash K_a \psi$. In either case $\Gamma \vdash K_a \psi$ holds. Also by the assumption of the lemma, $K_a \psi \in \Omega'$. These imply $K_a \psi \in \Gamma$ because Γ is an Ω'-saturated set.
($\Leftarrow$) Assume $K_a \in \Gamma$. There exists ψ' that does not begin with K_a such that $\psi = (K_a)^* \psi'$. By definition of f_a, $\psi' \in f_a(\Gamma)$. By induction hypothesis, $(f'_a(\Omega'), f_a(\Gamma)) \models \psi'$. This is equivalent to $M^{\mathrm{c}}(\Omega), (\Omega', \Gamma) \models K_a \psi'$. Since f_a^{c} is idempotent, $M^{\mathrm{c}}(\Omega), (\Omega', \Gamma) \models K_a \psi$.

(Case $\varphi = \psi_0 \supset \psi_1$) ($\Rightarrow$) Assume $M^{\mathrm{c}}(\Omega), (\Omega', \Gamma) \models \psi_0 \supset \psi_1$. Seeking contradiction, assume $\psi_0 \supset \psi_1 \notin \Gamma$. Since Γ is deductively closed, $\Gamma \cup \{\psi_0\} \not\vdash \psi_1$. By Lemma 35, there exists an Ω'-saturated set Γ' with $\Gamma' \supseteq \Gamma \cup \{\psi_0\}$ and $\Gamma' \not\vdash \psi_1$. By induction hypothesis, $M^{\mathrm{c}}(\Omega), (\Omega', \Gamma') \models \psi_0$ but not $M^{\mathrm{c}}(\Omega), (\Omega', \Gamma') \models \psi_1$. Since $(\Omega', \Gamma') \succeq (\Omega', \Gamma)$, this contradicts to $M^{\mathrm{c}}(\Omega), \Gamma \models \psi_0 \supset \psi_1$.
($\Leftarrow$) For an Ω'-saturated set Γ, assume $\psi_0 \supset \psi_1 \in \Gamma$. Take a state (Ω'', Δ) with $(\Omega', \Gamma) \preceq^{\mathrm{c}} (\Omega'', \Delta)$ and $M^{\mathrm{c}}(\Omega), (\Omega'', \Delta) \models \psi_0$. Showing $M^{\mathrm{c}}(\Omega), (\Omega'', \Delta) \models \psi_1$ is enough. By induction hypothesis, $\psi_0 \in \Delta$. Since $\psi_0 \supset \psi_1 \in \Gamma \subseteq \Delta$ and Δ is an Ω''-saturated set, $\psi_1 \in \Delta$. By induction hypothesis, $M^{\mathrm{c}}(\Omega), (\Omega'', \Delta) \models \psi_1$ holds.

□

Definition 39. *We define the length of a formula φ inductively on φ:*

$$\begin{aligned}
\mathrm{len}(\bot) = \mathrm{len}(P) &= 1, \\
\mathrm{len}(K_a \varphi) &= \mathrm{len}(\varphi) + 1, \\
\mathrm{len}(\varphi \land \psi) &= \mathrm{len}(\varphi) + \mathrm{len}(\psi) + 1, \\
\mathrm{len}(\varphi \lor \psi) &= \mathrm{len}(\varphi) + \mathrm{len}(\psi) + 1, \\
\mathrm{len}(\varphi \supset \psi) &= \mathrm{len}(\varphi) + \mathrm{len}(\psi) + 1.
\end{aligned}$$

Notation 40. *We denote a set of formulas which only contain the propositional variables in $V \subseteq PVar$ as* $\mathbf{Fml}_V$*.*

Lemma 41. *For a set of propositional variables $V \subseteq$ **Fml**, the length-limited set of formulas $\Gamma_n = \{\varphi \in \mathbf{Fml}_V \mid \mathrm{len}(\varphi) \leq n\}$ is hereditary f'-subformula-closed.*

Proof. By induction on n, we show a stronger proposition: both Γ_n and $\Gamma_n \cup K_a\Gamma_n$ are hereditary f'-subformula-closed for any $a \in A$.

(Case $n = 0$**)** Since $\Gamma_0 = \emptyset$, $\Gamma_0 \cup K_a\Gamma_0 = \emptyset$. Both are hereditary f'-subformula-closed.

(Case $n = n_0 + 1$**)** By definition of Γ_n and the definition of subformula, Γ_n is subformula-closed set. For any $a \in A$, since $g_a(\Gamma_n) = \Gamma_{n_0}$, $f'_a(\Gamma) = \Gamma_{n_0} \cup K_a\Gamma_{n_0} \subseteq \Gamma_n$ holds. Thus, by induction hypothesis, $f'_a(\Gamma)$ is a hereditary f'-subformula-closed set. These facts imply Γ_n is a hereditary f'-subformula-closed set.

We also show $\Gamma_n \cup K_b\Gamma_n$ is a hereditary f'-subformula-closed set. Since $\Gamma_n \cup K_b\Gamma_n$ is subformula-closed, we only have to check that $f'_c(\Gamma_n \cup K_b\Gamma_n)$ is f'-subformula-closed.

(Case $c \neq b$**)** $f'_c(\Gamma_n \cup K_b\Gamma_n) = f'_c(\Gamma_n)$, which is shown to be a hereditary f'-subformula-closed set.

(Case $c = b$**)** $f'_c(\Gamma_n \cup K_b\Gamma_n) = f'_c(f'_c(\Gamma_n)) = f'_c(\Gamma_n)$, which is shown to be a hereditary f'-subformula-closed set.

□

Theorem 42 (Finite model property)**.** *If φ is not a theorem of **IEC**, there is a finite model M with $M \not\models \varphi$.*

Proof. Since a formula φ is finitary, it contains only a finite number of propositional variables. Let V be the set of propositional variables occurring in φ. The set $\Omega = \{\psi \in \mathbf{Fml}_V \mid \mathrm{len}(\psi) \leq \mathrm{len}(\varphi)\}$ is finite and hereditary f'-subformula-closed by Lemma 41. By Lemma 35, there exist an Ω-saturated set Γ with $\varphi \notin \Gamma$. By Lemma 38, $M^{\mathrm{c}}(\Omega), (\Omega, \Gamma) \not\models \varphi$. Since Ω is finite, the model $M^{\mathrm{c}}(\Omega)$ is finite. In fact, the number of the states of $M^{\mathrm{c}}(\Omega)$ is at most $4^{|\Omega|}$. □

Theorem 43. *It is decidable whether a formula φ is a theorem of **IEC** or not.*

Proof. Since both proofs and finite models are recursively enumerable, the set of theorems and its complement are both recursively enumerable. □

5 Conclusion

We proved disjunction property and finite model property for an intuitionistic modal logic called intuitionistic epistemic logic (**IEC** for short). Disjunction property ensures that the intuitionistic epistemic logic is a constructive logic. Finite model property tells us it is unnecessary to consider non well-founded models, which does not seem to model any computation.

References

[1] Y. Hirai. An intuitionistic epistemic logic for sequential consistency on shared memory. *accepted to LPAR-16*, 2010.

[2] M. Sato. A study of Kripke-type models for some modal logics by Gentzen's sequential method. *Publications of the research institute for mathematical sciences*, 13:381, 1977.

[3] A.S. Troelstra and D. van Dalen. *Constructivism in Mathematics: An Introduction: Vol.: 1.* North-Holland, 1988.