DUALITY BETWEEN NONDETERMINISM AND CONCURRENCY ON GENERALIZED PETRI NET

ペトリネットの拡張上での非決定性と並列性の双対について

by

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ABSTRACT

We introduce a new state transition system based on Petri net, the generalized Petri net. C. A. Petri invented Petri net and introduced the duality of Petri net [1]. However, a marking on a net has no dual. On a generalized Petri net, the dual of a marking is a marking on the dual net, and the dual of a state transition is a state transition on the dual net. This higher duality is realized by the coherence relation on markings and the polarity of places and transitions. We construct a game semantics for the multiplicative additive fragment of linear logic (MALL) on generalized Petri nets. As possible applications of generalized Petri nets, we propose analysis of system-user interaction and visual representation of quantum system dynamics.

論文要旨

ペトリネットを基にした拡張ペトリネットという状態遷移系を導入する。ペトリネットを考案した C.A. ペトリがペトリネットの双対性を導入 [1] したが、マーキングの双対は無かった。拡張ペトリネットでは、マーキングの双対は双対ネットのマーキングであり、状態遷移の双対は双対ネットの状態遷移である。この高い双対性は、マーキング上の整合関係とプレースやトランジションの極性によって得られる。拡張ペトリネットを用いて線形論理の乗法的加法的断片の充満完全なゲーム意味論を作る。また応用可能性として、システムとユーザの相互関係の解析や、量子系の動態の視覚的表示を、提案する。

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Contents

Chapter 1

Introduction

1.1 Motivation

The motivation, as well as the guideline, for this work is the concept of duality. Duality enables us to investigate problems from different complementary sides, while keeping the notational and deductive cost of the investigation lower than double.

1.1.1 Nondeterminism and Concurrency

A `fork(2)` implementation copies something (concurrency), while a `fork(2)` user obtains an unpredictable return value (nondeterminism). This example suggests that nondeterminism and concurrency are two faces of the same coin. Is there formally definable duality between nondeterminism and concurrency?

1.1.2 Petri Net Duality

The duality concept of Petri nets appears in a paper by Petri [1]. We conjecture that Petri net duality represents the duality between nondeterminism and concurrency. The two Petri nets shown in Figure 1.1 are dual, and we consider these to represent the duality between concurrency and nondeterminism.

1.2 Contribution

The contributions of this paper are as follows:

1. A new state transition system, the generalized Petri net is presented with possible applications.

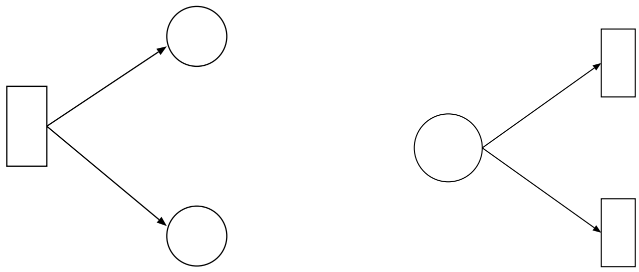

Figure 1.1: Two simple Petri nets showing concurrency (left) and nondeterminism (right).

   (a) A symmetric game semantics for MALL, the multiplicative additive fragment of linear logic, is presented and shown to be sound.
   (b) A visual representation of quantum system, which pays attention to the orthogonality between Hilbert subspaces, is presented.
   (c) Possible system modeling, which is aware of the duality between two sides of an interface, is presented.

2. The duality of Petri nets are extended:
   (a) The dual of a marking situation is a marking situation on the dual net.
   (b) The dual of a state transition is a state transition on the dual net.

1.3 Structure of this Paper

In Chapter 2 is preliminaries. In Chapter 3, we define the generalized Petri nets and investigate some of their properties. Our main result, the soundness of the generalized Petri net as a game semantics for multiplicative additive fragment of linear logic (MALL) appears in Chapter 4. In Chapter 5, two more adventurous possible applications are shown. We list related works in Chapter 6. We conclude in Chapter 7.

Chapter 2

Preliminaries

2.1 Petri Net

Definition 2.1.1. *A Petri net $N = \langle P, T, A, c\rangle$ is a tuple that consists of:*

1. *P, T, A: disjoint, finite sets.*
2. *$c : A \longrightarrow P \times T \cup T \times P$.*

Elements of P are called places, elements of T are transitions, and the elements of A are arcs of a Petri net $\langle P, T, A, c\rangle$. Places and transitions are called locations. We distinguish each arc to obtain a finer description of the Petri net dynamics. Also, we have a special denotation for entering arcs and leaving arcs of locations.

Notation 2.1.2. *On a Petri net $\langle P, T, A, c\rangle$, for a location $x \in P \cup T$, we let ${}^{\circ}x, x^{\circ} \subseteq A$ denote the sets of arcs defined as:*

$$
\begin{aligned}
{}^{\circ}x =& \{a \in A \mid c(a) = \langle s, x\rangle\} \\
x^{\circ} =& \{a \in A \mid c(a) = \langle x, d\rangle\}
\end{aligned}
$$

Definition 2.1.3. *A marking ($\langle M, \sigma\rangle$) of a Petri net $\langle P, T, -, -\rangle$ is a pair that consists of a finite set M and the locating function $\sigma : M \longrightarrow P$.*

Definition 2.1.4. *A firing relation between two markings $\langle M_0, \sigma_0\rangle[t\rangle\langle M_1, \sigma_1\rangle$ for a transition $t \in T$ holds when all of these conditions hold:*

1. *$\sigma_0 \restriction_{M_0 \cap M_1} = \sigma_1 \restriction_{M_0 \cap M_1}$,*
2. *A bijection $f : {}^{\circ}t \longrightarrow M_0 \setminus M_1$ exists and the equality $c(a) = \langle \sigma(f(a)), t\rangle$ holds for any $a \in {}^{\circ}t$, and*

3. *A bijection $g : t^\circ \longrightarrow M_1 \setminus M_0$ exists and the equality $c(a) = \langle t, \sigma(g(a))\rangle$ holds for any $a \in t^\circ$.*

Definition 2.1.5. *Marking sequence $(\langle M_i, \sigma_i\rangle)_{i\in I}$ (on I, a nonempty, successive sequence of integers) is a sequence of markings in which any successive two markings $\langle M_i, \sigma_i\rangle, \langle M_{i+1}, \sigma_{i+1}\rangle$ $(i, i+1 \in I)$ are in firing relation, that is, $\langle M_i, \sigma_i\rangle[t\rangle\langle M_{i+1}, \sigma_{i+1}\rangle$ for a transition $t \in T$.*

2.2 Coherence Space

Coherence space is an undirected graph in essence, and it provides a standard semantics for linear logic [2].

Definition 2.2.1 (Coherence Space)**.** *A coherence space $\langle M, \frown\!\!\!\!\smile\rangle$ is a pair of a set M and a symmetric, reflexive relation $\frown\!\!\!\!\smile$ on it.*

Given a coherence space $\langle M, \frown\!\!\!\!\smile\rangle$, some other relations are induced. Any one relation listed below with the base set M suffices to define a cohrence space.

notation	definition	pronunciation
$a \frown\!\!\!\!\smile b$		a and b are coherent
$a \frown b$	$a \frown\!\!\!\!\smile b$ and $a \neq b$	a and b are strictly coherent
$a \asymp b$	$a = b$ or $a \not\frown\!\!\!\!\smile b$	a and b are incoherent
$a \smile b$	$a \not\frown\!\!\!\!\smile b$	a and b are strictly incoherent

Table 2.1: Family of Coherence Relations

Note that the pair $\langle \frown\!\!\!\!\smile, \smile\rangle$ and the pair of $\langle \asymp, \frown\rangle$ are complementary: $M \times M = \frown\!\!\!\!\smile \sqcup \smile = \asymp \sqcup \frown$. Note also that these four relations are all symmetric, and the coherence $\frown\!\!\!\!\smile$ and the incoherence $\asymp$ are reflexive. The dual of a coherence space $\langle M, \frown\!\!\!\!\smile\rangle$ is the coherence space $\langle M, \asymp\rangle$.

In drawings like Figure 2.1, we use a blue solid line to represent coherence relation, and a red dashed line to represent incoherence relation throughout this paper. So, in Figure 2.1, these relations hold: $a \frown\!\!\!\!\smile b$, $a \frown b$, $a \frown\!\!\!\!\smile a$, $b \asymp c$, $b \smile c$, and $a \asymp a$.

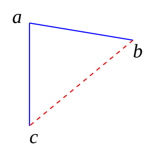

Figure 2.1: A coherence space consisting of three elements.

Chapter 3

Generalized Petri Nets

C. A. Petri [1] presented the duality of Petri nets. To investigate the duality further, we make a generalized concept of Petri nets. The extension allows tokens to reside not only on places but also on transitions. Also, a coherence relation introduced between tokens and polarity of places and transitions.

The duality for Petri nets have found few uses mainly because of the difficulty in defining the dual of a marked Petri net [3]. Tokens located in places find no location to reside on the dual Petri net, where places are turned into transitions. Since we allow tokens to reside on transitions, we can obtain the dual of marked Petri nets. In addition, we obtain the duality for a possible execution sequence of markings by adding polarities of places and transitions. The dual of an execution sequence is an execution sequence on the dual Petri net.

3.1 Definitions

3.1.1 Net

We add polarity for places and transitions.

Definition 3.1.1. *A generalized Petri net $N = \langle P, T, A, c, p\rangle$ is a tuple that consists of:*

1. *P, T, A: disjoint, finite sets.*

2. *$c : A \longrightarrow P \times T \cup T \times P$.*

3. *$p : P \cup T \longrightarrow \{+, -\}$, which satisfies the phasing condition described below.*

On a generalized Petri net, for a location $x \in P \cup T$, we let ${}^{\circ}x, x^{\circ} \subseteq A$ denote the sets of arcs *defined as:*

$$
\begin{aligned}
{}^{\circ}x &= \{a \in A \mid c(a) = \langle s, x\rangle\} \\
x^{\circ} &= \{a \in A \mid c(a) = \langle x, d\rangle\}
\end{aligned}
$$

Phasing Condition *The phasing condition imposed on the polarity p is that if two arcs $a, a' \in A$ start at the same location ($c(a) = \langle x, y\rangle$ and $c(a') = \langle x, z\rangle$) or end at the same location ($c(a) = \langle y, x\rangle$ and $c(a') = \langle z, x\rangle$) then the other locations y and z have the same polarity $p(y) = p(z)$.*

Elements of P are called places, elements of T are transitions, and elements of A are arcs of a generalized Petri net $\langle P, T, A, c, p\rangle$. Places and transitions are also called locations. The locations with the negative polarity – are called negative locations, while the locations with the positive polarity + are called positive locations.

Figure 3.1 is an example of a generalized Petri net.

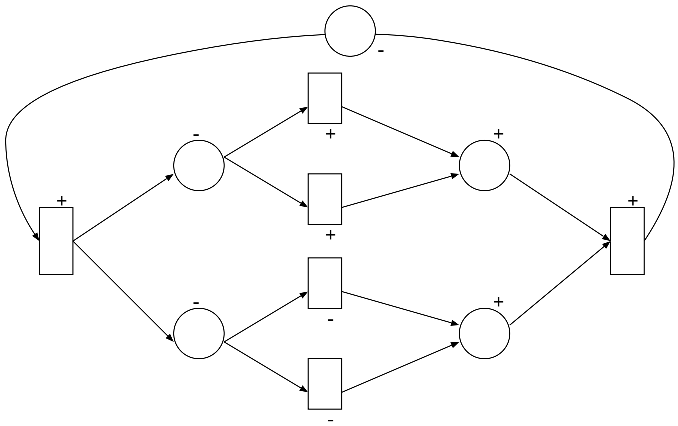

Figure 3.1: A generalized Petri net.

3.1.2 Marking Situation

Instead of having multisets as markings on a Petri net, we employ coherence spaces to denote a state of the generalized Petri net.

Definition 3.1.2. *A marking situation $\langle M, \sigma, \frown\rangle$ of a Petri net $\langle P, T, A, c, p\rangle$ is a tuple that consists of :*

- *$M, \frown$: a set and a relation on it that form a coherence space $\langle M, \frown\rangle$.*
- *$\sigma : M \longrightarrow P \cup T$.*

The elements of M are called tokens, the relation $\frown$ is called coherence, and the function σ is called the locating function of the marking situation.

3.1.3 Dynamics

Instead of having firing relation $M[t\rangle M'$ on Petri net markings, we have the evolution relation between marking situations.

Before defining the evolution relation, we introduce the synchronizing relation on arcs of a generalized Petri net. Note that the synchronizing relation on arcs defined below is static, only dependent on the net and not dependent of the marking situation.

Definition 3.1.3 (Synchronizing Relation on Arcs)**.** *Synchronizing relation $\approx$ is defined as the reflexive, transitive closure of the union $\frown_{\mathrm{c}} \cup \frown_{\mathrm{n}}$ of the concurrent coherence $\frown_{\mathrm{c}}$ on arcs and the nondeterministic coherence $\frown_{\mathrm{n}}$ on arcs, where*

- *Two arcs $a, a' \in A$ are concurrently coherent $a \frown_{\mathrm{c}} a'$ when the two arcs start at the same positive transition or end at the same positive transition.*
- *Two arcs $a, a' \in A$ are nondeterministically coherent $a \frown_{\mathrm{n}} a'$ when the two arcs start at the same negative place or end at the same negative place.*

On an evolution, tokens move along arcs like in firing of ordinary Petri nets. When a token moves along an arc $a \in A$, and there exists another arc a' which is in synchronizing relation with a ($a \approx a'$), then there must be another token that moves along the arc a'.

There are two kinds of arcs: the one kind of arcs flowing into transitions (absorptive); and the other kind of arcs flows out of transitions (emissive). We obtain a splitting $A = A_{P\to T} \sqcup A_{T\to P}$, where $A_{P\to T}$ denotes the set of absorptive arcs and $A_{T\to P}$ denote the set of emissive arcs. We find that the synchronizing relation preserves the splitting.

Proposition 3.1.4. *If $a \approx a'$ for two arcs $a, a' \in A$, either $a, a' \in A_{P\to T}$ or $a, a' \in A_{T\to P}$ holds.*

Proof. Since the relations $\frown_{\mathrm{c}}$ and $\frown_{\mathrm{n}}$ on A preserve the splitting, the union of both preserves it, and the reflexive transitive closure of the union also preserves it. □

This proposition enables us to define a natural splitting $A/\!\approx\; = (A_{P\to T}/\!\approx) \sqcup (A_{T\to P}/\!\approx)$. Likewise, the synchronizing relation $\approx$ preserves the four kinds of arcs defined by the polarities of their start points and their end points: $\langle +, +\rangle$, $\langle +, -\rangle$, $\langle -, +\rangle$, and $\langle -, -\rangle$.

Instead of firing relation on Petri nets, we have evolution relation. Just like a transition fires, an activation of a synchronized class of arcs (an element α of $A/\!\approx$) makes evolution. We give separate but dual definitions for two kinds of evolution, classified by whether the activated arcs are absorptive or emissive.

Definition 3.1.5 (absorptive evolution)**.** *For $\alpha \in A_{P\to T}/\approx$, two marking situations $\langle M,\sigma,\frown\rangle,\langle M',\sigma',\frown'\rangle$ and two functions $o : \alpha \longrightarrow M$, $n : \alpha \longrightarrow M'$, the absorptive evolution relation $\langle M,\sigma,\frown\rangle \xrightarrow{\langle\alpha,o,n\rangle} \langle M',\sigma',\frown'\rangle$ holds when the conditions described below are satisfied. In this definition, the set of consumed tokens $M \setminus M'$ is denoted by M_{old}, and the set of produced tokens $M' \setminus M$ is denoted by M_{new}.*

1. *For remaining tokens, their locations do not change, that is, $\sigma \restriction_{M\cap M'} = \sigma' \restriction_{M\cap M'}$ holds.*
2. *The functions o and n have specific codomains shown as $o : \alpha \longrightarrow M_{old}$ and $n : \alpha \longrightarrow M_{new}$. Also, they satisfy the conditions below. In these, the parent function $pa : M' \longrightarrow \wp(M)$ is defined as*

$$pa(m) = \begin{cases} \{m_{old} \in M_{old} \mid \langle o(a), n(a)\rangle = \langle m_{old}, m\rangle\} & (m \in M_{new}) \\ \{m\} & (m \in M \cap M') \end{cases}$$

   (a) $c(a) = \langle \sigma(o(a)), \sigma'(n(a))\rangle$ holds for any $a \in \alpha$.

   (b) For any $m_{old} \in M_{old}$ that is located at a positive place, there is only one $a \in \alpha$ that satisfies $o(a) = m_{old}$. Likewise, for any $m_{new} \in M_{new}$, that is located at a negative transition, there is only one $a \in \alpha$ that satisfies $n(a) = m_{new}$.

   (c) For any $m_{old} \in M_{old}$ that is located at a negative place p, the preimage $o^{-1}(m_{old}) = p^\circ$ and the restriction $n \restriction_{p^\circ}$ is an injection onto an $\asymp'$-clique. Likewise, for any $m_{new} \in M_{new}$ that is located at a positive transition t, the preimage $n^{-1}(m_{new}) = {}^\circ t$ and the restriction $o \restriction_{^\circ t}$ is injection onto a $\frown$-clique.

   (d) Two tokens $m'_0, m'_1 \in M'$ are incoherent if and only if there exist $m_0 \in pa(m'_0)$ and $m_1 \in pa(m'_1)$ that satisfies $m_0 \asymp m_1$.

Figure 3.2 shows an example of absorptive evolution.

Definition 3.1.6 (emissive evolution)**.** *For $\alpha \in A_{T\to P}/\approx$, two marking situations $\langle M,\sigma,\frown\rangle,\langle M',\sigma',\frown'\rangle$ and two functions $o : \alpha \longrightarrow M$, $n : \alpha \longrightarrow M'$, the emissive evolution relation $\langle M,\sigma,\frown\rangle \xrightarrow{\langle\alpha,o,n\rangle} \langle M',\sigma',\frown'\rangle$ holds when the conditions described below are satisfied. In this definition, the set of consumed tokens $M \setminus M'$ is denoted by M_{old}, and the set of produced tokens $M' \setminus M$ is denoted by M_{new}.*

1. *For remaining tokens, their locations do not change, that is, $\sigma \restriction_{M\cap M'} = \sigma' \restriction_{M\cap M'}$ holds.*
2. *The functions o and n have specific codomains shown as $o : \alpha \longrightarrow M_{old}$ and $n : \alpha \longrightarrow M_{new}$. Also, they satisfy the conditions below. In these, the parent function $pa : M' \longrightarrow$*

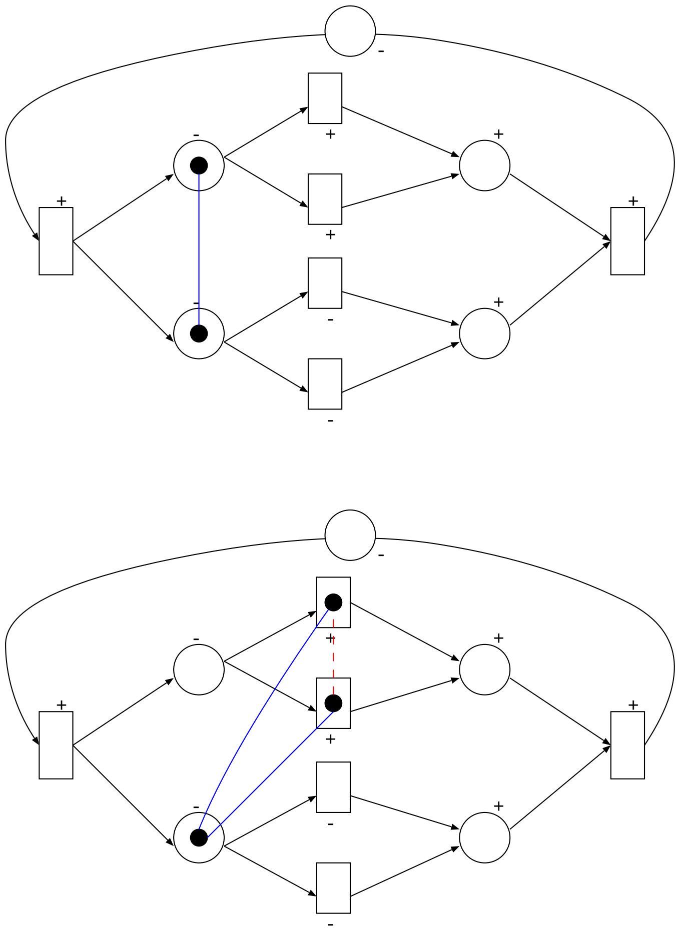

Figure 3.2: An example of absorptive evolution. The situation above becomes the situation below.

$\wp(M)$ is defined as

$$pa(m) = \begin{cases} \{m_{old} \in M_{old} \mid \langle o(a), n(a) \rangle = \langle m_{old}, m \rangle\} & (m \in M_{new}) \\ \{m\} & (m \in M \cap M') \end{cases}$$

(a) $c(a) = \langle \sigma(o(a)), \sigma'(n(a)) \rangle$ *holds for any* $a \in \alpha$.

(b) For any $m_{old} \in M_{old}$ *that is located at a negative transition, there is only one* $a \in \alpha$ *that satisfies* $o(a) = m_{old}$. *Likewise, for any* $m_{new} \in M_{new}$, *that is located at a positive place, there is only one* $a \in \alpha$ *that satisfies* $n(a) = m_{new}$.

(c) For any $m_{old} \in M_{old}$ *that is located at a positive transition* t, *the preimage* $o^{-1}(m_{old}) = x^{\circ}$ *and the restriction* $n \restriction_{t^{\circ}}$ *is injection onto a* $\smile'$*-clique. Likewise, for*

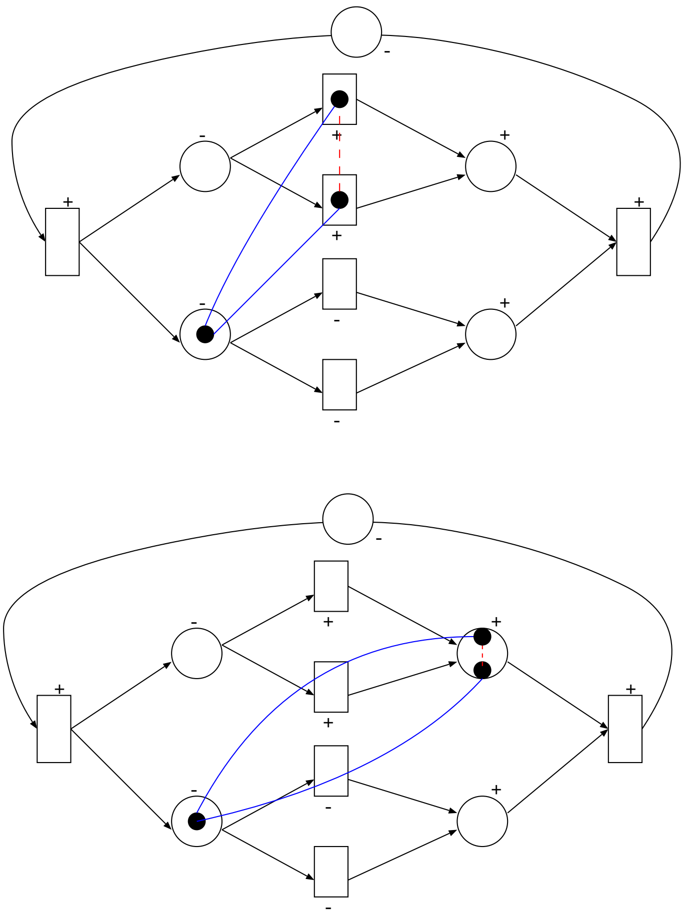

Figure 3.3: An example of emissive evolution. The situation above becomes the situation below after two steps of emissive evolutions. The tokens do not merge because the place they enter is positive.

any $m_{new} \in M_{new}$ that is located at a negative place p, the preimage $n^{-1}(m_{new}) = {}^{\circ}p$ and the restriction $o \restriction_{{}^{\circ}p}$ is injection onto an $\asymp$-clique.

(d) *Two tokens $m'_0, m'_1 \in M'$ are coherent if and only if there exist $m_0 \in pa(m'_0)$ and $m_1 \in pa(m'_1)$ that satisfies $m_0 \frown m_1$.*

Definition 3.1.7 (evolution)**.** *The evolution relation for $\alpha \in A/\approx$, two marking situations and two functions $o : \alpha \longrightarrow M$, $n : \alpha \longrightarrow M'$ is defined as the union of the absorptive evolution relation and the emissive evolution relation.*

Instead of having marking sequence as a possible history of markings, we introduce the concept of evolution sequence of marking situations.

Definition 3.1.8 (evolution sequence)**.** *An evolution sequence*

$$E = \langle M_0, \sigma_0, \frown_0 \rangle \xrightarrow{\langle \alpha_0, o_0, n_0 \rangle} \cdots \xrightarrow{\langle \alpha_{m-1}, o_{m-1}, n_{m-1} \rangle} \langle M_m, \sigma_m, \frown_m \rangle$$

on a generalized Petri net $\langle P, T, A, c, p \rangle$ is a sequence of marking situations connected by evolution relations.

When we need not explicitly denote α, o or n, we write an evolution sequence as

$$E = (\langle M_i, \sigma_i, \frown_i \rangle)_{i \in I}.$$

3.2 Properties

3.2.1 Duality

Definition 3.2.1. *The dual of the generalized Petri net $N = \langle P, T, A, c, p \rangle$ is a generalized Petri net $N^{\perp} = \langle T, P, A, , c, -p \rangle$, where the polarity $-p$ is defined as:*

$$-p(x) = \begin{cases} - & (p(x) = +) \\ + & (p(x) = -). \end{cases}$$

The conditions for a generalized Petri net is preserved by taking this dual.

Definition 3.2.2. *The dual of the evolution sequence $E = \langle M_0, \sigma_0, \frown_0 \rangle \xrightarrow{\alpha_0, o_0, n_0} \cdots \xrightarrow{\alpha_{m-1}, o_{m-1}, n_{m-1}} \langle M_m, \sigma_m, \frown_m \rangle$ is defined as $E^{\perp} = \langle M_0, \sigma_0, \asymp_0 \rangle \xrightarrow{\alpha_0, o_0, n_0} \cdots \xrightarrow{\alpha_{m-1}, o_{m-1}, n_{m-1}} \langle M_m, \sigma_m, \asymp_m \rangle$.*

The main property of the generalized Petri net is that the dual of a evolution sequence on a net N is a marking situation evolution sequence of the dual net $N^{\perp}$.

Theorem 3.2.3. *When E is an evolution sequence on a generalized Petri net N, the dual $E^{\perp}$ is a marking situation evolution sequence on the dual net $N^{\perp}$.*

Proof. The definition 3.1.5 of the evolution relation is invariant when we swap the terms shown below at the same time:

$$\text{coherence} \longleftrightarrow \text{incoherence}$$
$$\text{place} \longleftrightarrow \text{transition}$$
$$\text{positive} \longleftrightarrow \text{negative}.$$

Thus, each successive pair of marking situation in the evolution sequence $E^{\perp}$ is in evolution relation on the dual net $N^{\perp}$. □

3.2.2 Positive Nets are Similar to Ordinary Petri Nets

Definition 3.2.4. *A positive net is a generalized Petri net whose locations are all positive.*

In a positive net, nothing happens for incoherence. Dynamics of a positive net is that of an ordinary Petri net, with $\asymp$-cliques as markings and evolution as partial firing: the firing in the ordinary Petri nets takes two evolution steps, namely absorption (where tokens enter transitions) and emission (where tokens leave transitions).

3.2.3 Diffusive Nets are Locally Confluent

Definition 3.2.5. *A diffusive net is a generalized Petri net whose places are all negative and whose transitions are all positive.*

Theorem 3.2.6 (local confluence of diffusive nets)**.** *If* $\langle M_0, \sigma_0, \asymp_0 \rangle \xrightarrow{\alpha_1, o_1, n_1} \langle M_1, \sigma_1, \asymp_1 \rangle$ *and* $\langle M_0, \sigma_0, \asymp_0 \rangle \xrightarrow{\alpha_2, o_2, n_2} \langle M_2, \sigma_2, \asymp_2 \rangle$ *hold for different* α_1 *and* α_2 *on a diffusive net, there is another marking situation* $\langle M_3, \sigma_3, \asymp_3 \rangle$ *that satisfies* $\langle M_1, \sigma_1, \asymp_1 \rangle \xrightarrow{\alpha_2, o_2, n_2} \langle M_3, \sigma_3, \asymp_3 \rangle$ *and* $\langle M_2, \sigma_2, \asymp_2 \rangle \xrightarrow{\alpha_1, o_1, n_1} \langle M_3, \sigma_3, \asymp_3 \rangle$*.*

Proof. This follows easily from the fact that the sets α_1 and α_2 are disjoint and that the images $o_1(\alpha_1)$ and $o_2(\alpha_2)$ are disjoint. □

3.2.4 Focusing Nets Preserve Number of Tokens

Definition 3.2.7. *A focusing net is a generalized Petri net whose places are all positive and whose transitions are all negative.*

On a focusing net, the number of tokens never changes in an evolution. Individual tokens just move along arcs. No interaction occurs between tokens.

3.3 Timeline and Evolution Structure

In an evolution sequence, a pair of evolutions are chronically ordered when some tokens produced by an evolution is consumed by another evolution. When such intermediate tokens do not exist, swapping the two evolution does not hurt the validity of the marking situation evolution sequence nor does change what happens to individual tokens. When we consider the evolution sequences before and after the swap of evolutions as equivalent, we make an application of "true concurrency" concept in the realm of process algebras (Chaprter 2, [4]), which considers the sequence of global states as redundant and considers the partially ordered set of events as essential.

The truly concurrent representation of an evolution sequence is the evolution structure. The evolution structure is a directed graph whose vertices are tokens that ever appear in the marking situation evolution sequence and whose edges are occurrences of arcs as elements of α_i in the evolution sequence.

Below, we assume that if a token m appears in M_h and M_j, then it also appears in the intermediate marking situation M_i if $h \leq i \leq j$ in any evolution sequence $(\langle M_i, \sigma_i, \circlearrowleft_i \rangle)_{i \in I}$.

3.3.1 Construction

Definition 3.3.1 (pre-evolution structure)**.** *For an evolution sequence* $\langle M_0, \sigma_0, \circlearrowleft_0 \rangle \xrightarrow{\langle \alpha_0, o_0, n_0 \rangle} \cdots \xrightarrow{\langle \alpha_{m-1}, \sigma_{m-1}, \circlearrowleft_{m-1} \rangle} \langle M_m, \sigma_m, \circlearrowleft_m \rangle$*, the pre-evolution structure is a tuple* $\langle M, \sigma, E, o, n, \circlearrowleft, \asymp \rangle$*, where*

- *(the set of tokens)* $M = \bigcup_{i \in \{0,\dots,n\}} M_i$
- $\sigma = \bigcup_{i \in \{0,\dots,n\}} \sigma_i$
- *(the set of events)* $E = \{\langle a, i \rangle \mid a \in \alpha_i$ *for an* $i \in \{0, \dots, n-1\}\}$
- $o : E \longrightarrow M$ *is defined as* $o(\langle a, i \rangle) = o_i(a)$
- $n : E \longrightarrow M$ *is defined as* $n(\langle a, i \rangle) = n_i(a)$
- $\circlearrowleft = \bigcup_{i \in \{0,\dots,n\}} \circlearrowleft_i$
- $\asymp = \bigcup_{i \in \{0,\dots,n\}} \asymp_i$

Note that σ is still a function.

When there is an event e with $o(e) = m_{\mathrm{o}}$ and $n(e) = m_{\mathrm{n}}$, then we say m_{o} is a parent of m_{n} ($m_{\mathrm{o}} \rightarrowtail m_{\mathrm{n}}$). The reflexive, transitive closure of $\rightarrowtail$ forms a partial order $\preccurlyeq$ called the chronic precedence.

Lemma 3.3.2. *If* $l \rightarrowtail m$ *holds,* $l \preccurlyeq n \preccurlyeq m \Longrightarrow l = n$ *or* $n = m$ *holds.*

Proof. When $l \neq n$ and $l \preccurlyeq n$ hold, this means that n is a descendant of l, thus, n appears at the same as or after l disappears in the evolution sequence. Likewise, when $n \neq m$ and $n \preccurlyeq m$ hold, this means that m is a descendant of n, thus, m appears at the same time as or after n disappears in the evolution sequence. Combining these two incurs contradiction. □

Notation 3.3.3. *For a token* m *in a pre-evolution structure,* $m\downarrow$ *denote the set of tokens* $\{m' \mid m' \preccurlyeq m\}$*. Likewise,* $m\uparrow$ *denote the set of tokens* $\{m' \mid m \preccurlyeq m'\}$*.*

An indispensable part of truly concurrent representation of marking sequence evolution is timeline, a chain of parent-child tokens.

Definition 3.3.4 (timeline)**.** *A timeline $T = (m_i)_{i \in I}$ in a pre-evolution structure is a sequence of tokens in which successive tokens are in parent-child relation, that is, $m_i \rightarrowtail m_{i+1}$ holds for any $i, i+1 \in I$. Also, a timeline must start with a minimal token and end with a maximal token in the ordered set $\langle M, \preccurlyeq \rangle$.*

We say that m_0 is an ancestor of m_1 and that m_1 is a descendant of m_0 when $m_0 \preccurlyeq m_1$ for different two tokens ($m_0 \neq m_1$), this is denoted $m_0 \prec s_1$.

Definition 3.3.5. *The cooccurrence relation C on the pre-evolution structure $\langle M, -, -, -, -, \supset, \asymp \rangle$ is defined as the union of $\supset$ and $\asymp$:*

$$C = \supset \cup \asymp .$$

The name of the cooccurrence relation C comes from the fact that $\langle m_0, m_1 \rangle \in C$ holds if and only if both m_0 and m_1 are contained in a marking situation M_i of the evolution sequence $(\langle M_i, \sigma_i, \supset_i \rangle)_{i \in I}$ on which the pre-evolution structure is based.

For a token m, we denote the set of marking situations containing m as $\bar{m} \subset I$:

$$\bar{m} = \{ i \in I \,|\, m \in M_i \},$$

where M_i is a marking situation appearing in the evolution sequence $E = (\langle M_i, \sigma_i, \supset_i \rangle)_{i \in I}$. By the construction of the pre-evolution structure, for any token m, $\bar{m}$ is not empty. Also, for any index $i \in I$, any timeline T has a token $m \in T$ with $i \in \bar{m}$. Actually, the reason for the name timeline is the fact that a timeline T induces a splitting of I, the index set of the evolution sequence:

$$I = \bigsqcup_{m \in T} \bar{m},$$

where each $\bar{m}$ is a nonempty set of successive integers, and when $m \prec m'$ holds, every element of $\bar{t}$ is smaller than any element of $\bar{t}'$.

Note also that $\langle m, m' \rangle \in C \iff \bar{m} \cap \bar{m}' \neq \varnothing$ holds.

Proposition 3.3.6 (timeline and cooccurrence)**.** *Two timelines T_0, T_1 and the cooccurrence relation C based on an evolution sequence E has the following two properties:*

1. *For any $m_0 \in T_0$, there exists $m_1 \in T_1$ and the two are in cooccurrence relation $\langle m_0, m_1 \rangle \in C$.*

2. *For any $m_0, m_0' \in T_0$ and $m_1, m_1' \in T_1$ satisfying $\langle m_0, m_1\rangle, \langle m_0', m_1'\rangle \in C$, at least one of the two conjunctions below holds:*

$$m_0 \preccurlyeq m_0' \text{ and } m_1 \preccurlyeq m_1', \text{ or}$$

$$m_0' \preccurlyeq m_0 \text{ and } m_1' \preccurlyeq m_1$$

Proof. Since $\bar{m}_0$ is nonempty, we can take any index $i \in \bar{m}_0$ and then find $m_1 \in T_1$ that satisfies $i \in \bar{m}_1$

Since $\langle m_0, m_1\rangle \in C$ and $\langle m_0', m_1'\rangle \in C$ hold, both intersections $\bar{m}_0 \cap \bar{m}_1$ and $\bar{m}_0' \cap \bar{m}_1'$ are nonempty successive set of integers. When the intersection of the two intersections $(\bar{m}_0 \cap \bar{m}_0') \cap (\bar{m}_1 \cap \bar{m}_1')$ is not empty, all of the four relations $m_0 \preccurlyeq m_0'$, $m_1 \preccurlyeq m_1'$, $m_0' \preccurlyeq m_0$ and $m_1' \preccurlyeq m_1$ hold because $m_0 = m_0'$ and $m_1 = m_1'$ hold. Otherwise when the intersection of the two intersections $(\bar{m}_0 \cap \bar{m}_0') \cap (\bar{m}_1 \cap \bar{m}_1')$ is empty, all elements of on intersection is chronically precedent in comparison with any element of the other intersection. This time, only one of the two conjunctions holds. $\square$

The structure defined is not yet an evolution structure because there possibly exist two tokens $m, m' \in M$ that are not coherent, incoherent nor chronically ordered. A typical example is the tokens m_0 and m_1' in the evolution sequence $\langle\{m_0, m_1\}, -, -\rangle \to \langle\{m_0', m_1\}, -, -\rangle \to \langle\{m_0', m_1'\}, -, -\rangle$. Although m_1' appears after m_0 disappears, these tokens are not chronically ordered because no token produced by the disappearance of m_0 is consumed when m_1' appears.

The evolution structure is obtained by adding coherence, incoherence and chronic precedence by interpolation. The interpolation is possible by the following lemma.

Lemma 3.3.7. *For an pre-evolution structure and two tokens m_0 and m_1 which are not chronically ordered, when we define*

$$D_0 = C \cap (m_0{\downarrow} \times m_1{\uparrow}) \qquad \text{and} \qquad D_1 = C \cap (m_0{\uparrow} \times m_1{\downarrow}),$$

one of the two conditions below holds:

1. *$D_0 \cap D_1$ is a singleton set $\{\langle m_0, m_1\rangle\}$.*

2. *One of the two sets D_0 and D_1 is empty, and the other is nonempty.*

Proof. If $D_0 \cap D_1$ is nonempty, the condition 1. holds because the only common element of the sets $(m_0{\downarrow} \times m_1{\uparrow})$ and $(m_0{\uparrow} \times m_1{\downarrow})$ is $\langle m_0, m_1\rangle$, and the condition 2. does not hold because both D_0 and D_1 are nonempty.

Otherwise if $D_0 \cap D_1$ is empty, the condition 2. holds. Either D_0 or D_1 is nonempty because, by the statement 1. of Lemma 3.3.6, there is a pair $\langle m_0, m_1'\rangle \in C$, which is also in

$C \cap (m_0{\downarrow} \times m_1{\uparrow}) \cup (m_0{\uparrow} \times m_1{\downarrow})$, making $D_0 \cup D_1$ nonempty. Either D_0 or D_1 is empty because if both are nonempty, they have elements $\langle t_0, t_0' \rangle \in D_0$ and $\langle t_1, t_1' \rangle \in D_1$. If $t_0 = t_1 = m_0$, then $\langle m_0, m_1 \rangle \in C$ and $D_0 \cap D_1 \neq \varnothing$ follows. Since at least one of t_0 and t_1 is not m_0 and at least one of t_0' and t_1' is not m_1, the chronic precedence of these elements contradicts the statement 2. of Lemma 3.3.6. Since one of the sets is empty, the condition 1. does not hold. □

On top of a given pre-evolution structure, we define new relations $\overset{\sim}{\substack{\frown\\\smile}}$, $\overset{\sim}{\asymp}$, and $\overset{\sim}{\preccurlyeq}$. They extend $\substack{\frown\\\smile}$, $\asymp$, and $\preccurlyeq$ respectively.

Definition 3.3.8. *For two tokens m_0 and m_1, if the two tokens are chronically ordered, they are in $\overset{\sim}{\preccurlyeq}$ relation accordingly:*

$$m_0 \preccurlyeq m_1 \Longrightarrow m_0 \overset{\sim}{\preccurlyeq} m_1, \qquad m_1 \preccurlyeq m_0 \Longrightarrow m_1 \overset{\sim}{\preccurlyeq} m_0.$$

If they are not chronically ordered, they are automatically not equal. One of the two conditions stated in Lemma 3.3.7 holds. If the condition 1. of Lemma 3.3.7 holds, the two tokens are coherent or incoherent exclusively. Then, they are in the extended (in)coherence relation accordingly:

$$m_0 \mathrel{\substack{\frown\\\smile}} m_1 \Longrightarrow m_0 \mathrel{\overset{\sim}{\substack{\frown\\\smile}}} m_1, \qquad m_0 \asymp m_1 \Longrightarrow m_0 \overset{\sim}{\asymp} m_1.$$

Otherwise if 2. holds and D_0 is not empty, we define an order $\gtrless$ on D_0. The pair $\langle a, b \rangle \in D_0$ is less than the pair $\langle c, d \rangle \in D_0$ (denoted $\langle a, b \rangle \gtrless \langle c, d \rangle$) if and only if both $c \preccurlyeq a$ and $b \preccurlyeq d$ hold. We let $m_0 \mathrel{\overset{\sim}{\substack{\frown\\\smile}}} m_1$ hold when all minimal values in $\langle D_0, \gtrless \rangle$ are pairs of coherent tokens. We let $m_0 \overset{\sim}{\asymp} m_1$ hold when all minimal values in $\langle D_0, \gtrless \rangle$ are pairs of incoherent tokens. If none of the above holds, we let $m_1 \overset{\sim}{\preccurlyeq} m_0$ hold.

Likewise, if 2. holds and D_1 is not empty, we define an order $\lessgtr$ on D_1. The pair $\langle a, b \rangle \in D_1$ is less than the pair $\langle c, d \rangle \in D_1$ (denoted $\langle a, b \rangle \lessgtr \langle c, d \rangle$) if and only if both $a \preccurlyeq c$ and $d \preccurlyeq b$ hold. We let $m_0 \mathrel{\overset{\sim}{\substack{\frown\\\smile}}} m_1$ hold when all minimal values in $\langle D_1, \lessgtr \rangle$ are pairs of coherent tokens. We let $m_0 \overset{\sim}{\asymp} m_1$ hold when all minimal values in $\langle D_1, \lessgtr \rangle$ are pairs of incoherent tokens. If none of the above holds, we let $m_0 \overset{\sim}{\preccurlyeq} m_1$ hold.

The extended chronic precedence $\overset{\sim}{\preccurlyeq}$ satisfies this property.

Proposition 3.3.9. *The extended chronic precedence $\overset{\sim}{\preccurlyeq}$ has the following properties with regard to the cooccurrence relation C:*

1. *If $l \overset{\sim}{\preccurlyeq} m$, there exists m^- that satisfies both $m^- \preccurlyeq m$ and $\langle l, m^- \rangle \in C$.*
2. *Likewise, if $m \overset{\sim}{\preccurlyeq} n$, there exists m^+ that satisfies both $m \preccurlyeq m^+$ and $\langle n, m^+ \rangle \in C$.*

Proof. 1. Assume $l \overset{\sim}{\preccurlyeq} m$. There are three possible reasons for the extended precedence.

(a) When $l \preccurlyeq m$ holds, we can take $m^{-} = l$ then the statement holds.

(b) When there are more than one minimal values for $\langle D_0, \gtrless \rangle$, where $D_0 = C \cap (m\!\downarrow \times l\!\uparrow)$, the nonemptiness of D_0 and the condition 1. and 2. of Proposition 3.3.6 yields the existence of m^{-}.

(c) The other case where there are more than one minimal values for $\langle D_1, \lessgtr \rangle$ is in essence the same as the case 1b above this.

2. Similar.

□

The extended chronic precedence $\tilde{\preccurlyeq}$ has reflexivity and antisymmetri, however, the exact condition for the extended precedence $\tilde{\preccurlyeq}$ to have transitivity is not yet found. Using these extended relations, we define the evolution structure.

Definition 3.3.10. *The evolution structure based on a pre-evolution structure is the tuple* $\langle M, \sigma, E, o, n, \tilde{\frown}, \tilde{\asymp}, \tilde{\preccurlyeq} \rangle$.

3.3.2 Properties

Proposition 3.3.11. *An evolution structure* $\langle M, \sigma, E, o, n, \tilde{\frown}, \tilde{\asymp}, \tilde{\preccurlyeq} \rangle$ *on a generalized Petri net* $\langle P, T, A, c, p \rangle$ *satisfies all conditions described below. Elements of M are called tokens, and elements of E are called events. A pair of tokens $\langle m, m' \rangle$ satisfying $m = o(e)$ and $m' = n(e)$ for some $e \in E$ is called a parent-child pair. A subset M' of M is called synchronic when either $m' \tilde{\frown} m''$ or $m' \tilde{\asymp} m''$ holds for any $m', m'' \in M'$.*

1. *(integrity) For any two different tokens $m, m' \in M$, only and exactly one of the following holds: $m \tilde{\preccurlyeq} m'$, $m' \tilde{\preccurlyeq} m$, $m \tilde{\frown} m'$, or $m \tilde{\asymp} m'$.*

2. *(chronic atomicity of events) For a parent-child pair $\langle l, n \rangle$, there is no token $m \in M$ that satisfies the relation $l \tilde{\preccurlyeq} m \tilde{\preccurlyeq} n$ although m is neither synchronic with l nor n.*

3. *(event-coherence interference) For any three tokens m, m_0 and m_1 with $m \neq m_0$ and $m \neq m_1$, if either $\langle m_0, m_1 \rangle$ or $\langle m_1, m_0 \rangle$ is a parent-child pair and $\sigma(m_0)$ is a place, then, both of these hold:*

$$m_0 \tilde{\asymp} m \Longrightarrow m_1 \tilde{\asymp} m, \qquad m_1 \tilde{\frown} m \Longrightarrow m_0 \tilde{\frown} m.$$

4. *(types of branch) If two different events share a start (resp. end) token, the end (resp. start) tokens of the events are coherent if the shared start (resp. end) token is located in a transition, otherwise incoherent.*

5. *(simpleness) If two events $\langle a, i\rangle$ and $\langle a, j\rangle$ share an arc a and a start (resp. end) token $o(\langle a, i\rangle) = o(\langle a, j\rangle)$ (resp. $n(\langle a, i\rangle) = n(\langle a, j\rangle)$), the two events are identical.*

Proof. (integrity) Follows from the Definition 3.3.8.
(chronic atomicity of events) Assume $l \rightarrowtail n$ and $l \mathrel{\tilde{\lessapprox}} m \mathrel{\tilde{\lessapprox}} n$ hold. By Proposition 3.3.9 there are two tokens m^- and m^+ that satisfy $m^- \preccurlyeq m \preccurlyeq m^+$ and $\langle l, m^-\rangle, \langle m^+, n\rangle \in C$. There is a timeline T_m containing m^-, m and m^+ in this order. There is another timeline T containing l and n successively. By Proposition 3.3.6, either $\langle m, l\rangle$ or $\langle m, n\rangle$ is an element of C.
(event-coherence interference) This follows from the condition 2d of the Definition 3.1.5.
(types of branch) This follows from the conditions 2b, 2c and 2d of the Definition 3.1.5.
(simpleness) This follows from the conditions 2b and 2c of the Definition 3.1.5. □

3.3.3 Evolution Structure Classifies Evolution Sequences

The construction of the evolution structure from an evolution sequence is unique, but not an injection, so some information is thrown away. When different evolution structure yields the same evolution structure up to isomorphism, the order of events that are not chronically ordered is discarded.

Chapter 4

Game Semantics for MALL

Using generalized Petri nets, we can easily obtain a game semantics for the multiplicative additive fragment of linear logic (MALL), which is a special case of the game defined in [5]. A net with initial marking situation represents a formula of MALL, and a proof for the formula is represented by a winning strategy for the game. The game board is a direct syntactic translation of the linear formula. The two kinds of places represent two kinds of disjunction, while the two kinds of transitions represent two kinds of conjunction.

4.1 MALL

The multiplicative additive fragment of linear logic (MALL) lacks the two modalities $!, ?$ and the four units $0, \top, 1$ and $\bot$ of the full propositional linear logic, but it has all the linear connectives $\otimes, \oplus, \parr$ and &.

4.1.1 Syntax

Atomic Formulae We assume a countably infinite set U of atomic formulae. We also assume that there is an involution $(-)^{\perp} : U \longrightarrow U$ which satisfies $X^{\perp} \neq X$ and $(X^{\perp})^{\perp} = X$ for any $X \in U$. Also, we introduce a polarity in U. Some elements of U are positive while the others are negative. When X is positive, $X^{\perp}$ is negative, and vice versa.

MALL Formulae The logical formulae of MALL is defined recursively as

$$F, G ::= X \,|\, X^{\perp} \,|\, F \otimes G \,|\, F \oplus G \,|\, F \parr G \,|\, F \,\&\, G.$$

The formula X denotes a positive atomic formula in U. And the formula $X^{\perp}$ denotes a negative atomic formula in U. Other forms of formulae are called composite formulae. There are two kinds of composite formulae: one is synchronous formulae of the forms $F \otimes G$ and

$F \oplus G$; the other is asynchronous formulae of the forms $F \& G$ and $F \parr G$. Note that $(-)^\perp$ is only defined on atomic formulae but not on composite formulae for simplicity.

As an abbreviation, negation $(-)^\perp$ of composite formulae are defined recursively using De Morgan's dualities as follows:

$$(F \otimes G)^\perp := (F^\perp \parr G^\perp),$$
$$(F \oplus G)^\perp := (F^\perp \& G^\perp),$$
$$(F \parr G)^\perp := (F^\perp \otimes G^\perp),$$
$$(F \& G)^\perp := (F^\perp \oplus G^\perp).$$

4.1.2 Proof System: Focusing Proof

The proof system described here is devised by Andreoli in [6]. It is shown to be sound and complete for the standard sequent calculus of MALL.

In the figures below, F, G stand for formulae, X stands for a positive atom, Γ and Δ stand for multisets of formulae containing no asynchronous formula. Θ stands for multisets of formulae in general.

- Asynchronous phase

$$\frac{\vdash \Gamma \Uparrow \Theta, F, G}{\vdash \Gamma \Uparrow \Theta, F \parr G} \ (\parr)$$

$$\frac{\vdash \Gamma \Uparrow \Theta, F \qquad \vdash \Gamma \Uparrow \Theta, G}{\vdash \Gamma \Uparrow \Theta, F \& G} \ (\&)$$

$$\frac{\vdash \Gamma, F \Uparrow \Theta}{\vdash \Gamma \Uparrow \Theta, F} \ (R \Uparrow) \text{ (if } F \text{ is not asynchronous)}$$

- Synchronous phase

$$\frac{\vdash \Gamma \Downarrow F \qquad \vdash \Delta \Downarrow G}{\vdash \Gamma, \Delta \Downarrow F \otimes G} \ (\otimes)$$

$$\frac{\vdash \Gamma \Downarrow F}{\vdash \Gamma \Downarrow F \oplus G} \ (\oplus_l) \qquad \frac{\vdash \Gamma \Downarrow G}{\vdash \Gamma \Downarrow F \oplus G} \ (\oplus_r)$$

$$\frac{\vdash \Gamma \Uparrow F}{\vdash \Gamma \Downarrow F} \ (R \Downarrow) \text{ (if } F \text{ is not synchronous)}$$

- Identity and Decision

$$\frac{}{\vdash X \Downarrow X^\perp} \ (I) \qquad \frac{\vdash \Gamma \Downarrow F}{\vdash \Gamma, F \Uparrow} \ (D)$$

A formula F is proved when the sequent $\vdash F \Uparrow$ is deduced.

4.2 Linear Petri Net Game

The linear Petri net game is a game played by two players. Throughout a play, a static game board is fixed, while the game position changes as players make moves.

4.2.1 Game Board

A game board $B = \langle P, T, A, c, p \rangle$ is a generalized Petri net.

4.2.2 Game Position

A game position $\langle M, \frown, \sigma, \tau, S \rangle$ is a marking situation $\langle M, \frown, \sigma \rangle$ with a turn $\tau \in \{\mathrm{P}, \mathrm{O}\}$ and a set $S \subset M$ of discarded tokens.

The dual position of a position $\langle M, \frown, \sigma, \tau, S \rangle$ is a game position $\langle M, \asymp, \sigma, -\tau, S \rangle$, in which the coherence space and the turn are taken dual but the locating function σ and the set of discarded tokens remain the same.

4.2.3 Game

A game $\langle B, P_{\mathrm{init}} \rangle$ is a pair of a game board B and an initial game position P_{init} on it. The initial game position P_{init} has only one token and an empty discarded token set. If the single token is at a positive location, the first turn is Player's. Otherwise, the first turn is Opponent's.

4.2.4 Move

A Player's move is a sequence of evolutions that satisfies the three conditions below:

1. There exist game positions $(P_i)_{\{}i \in \{0, \ldots, n\}\}$ that that satisfies $P_i \rightarrow^{s_i} P_{i+1}$ for all $0 \leq i \leq n-1$.

2. All consumed tokens are located at a positive location.

3. All produced tokens are either located at a negative location or consumed by another absorption or emission.

4. All evolutions except the first one consumes only tokens produced by anothetr evolution in the move.

When no play is available to Player, Player can pass. An Opponent's move is an evolution that consumes a token in a negative location. Opponent can pass when no move is available to him.

4.2.5 Winning Condition

No general winning condition for Linear Petri Net Game is defined yet. To model MALL, we devised a special winning condition described in the next section.

4.3 From MALL Formula to Game

For a given MALL formula, we can construct the corresponding Linear Petri Net Game as follows:

1. Construct a directed graph whose vertices are sub-formulae of the given formula.
2. Replace the vertices with polarized petri net nodes.
3. Add some supplementary linear petri net nodes and set initial position.

4.3.1 Graph Construction

The directed graph $\langle V, E, c\rangle$ ($c : E \longrightarrow V \times V$) is constructed as follows.

The set of sub-formulae is the set of vertices V. Even if there are syntactically equivalent sub-formulae, each ocurrance is treated as an element of V. From a composite sub-formula $F \diamond G$, two edges $e_1, e_2 \in E$ are drawn to F and to G. The locating function c locates the two edges as $c(e_1) = \langle F \diamond G, F\rangle$ and $c(e_2) = \langle F \diamond G, G\rangle$.

The graph $\langle V, E, c\rangle$ is a rooted tree by construction.

4.3.2 Node Replacement

The vertices are split into places and transitions $V = P \sqcup T$ and given polarity $\pi : V \longrightarrow \{-1, 1\}$ following these rules.

- A negative atom is a negative place.
- A positive atom is a positive transition.
- A composite formula is
  - a negative place if the topmost connective is $⅋$,
  - a negative transition if the topmost connective is &,
  - a positive transition if the topmost connective is $\otimes$, or
  - a positive place if the topmost connective is $\oplus$.

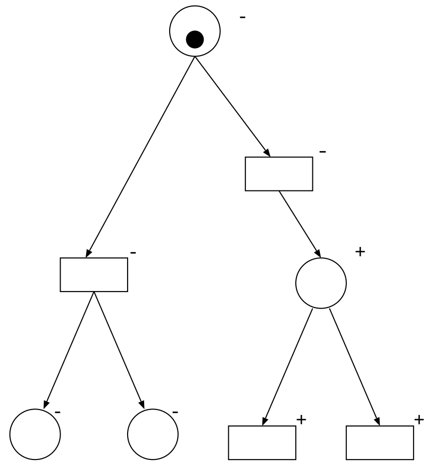

Figure 4.1: The game $[\![(A \oplus B) \parr (A^{\perp} \,\&\, B^{\perp})]\!]$

.

These rules translate dual pairs of linear connectives to dual pairs of game locations. We denote an element of V by the expression of the corresponding subformula when no confusion is expected.

4.3.3 Supplementary Locations and Initial Position

The definition of constructed tree contains arcs between two places and arcs between transitions. Since the definition of generalized Petri net does not allow such arcs, we add intermediate locations for such arcs.

An edge $e \in E$ from a transition t_0 to a transition t_1 is split into two arcs a_0 and a_1. An intermediate place p called a glue is introduced and used as the ending location of a_0 as well as the starting location of a_1. The polarity of the glue p is the same as that of t_0. Edges connecting two places from p_0 to p_1 are treated like this. Again this time, the polarity of the intermediate transition is the same as that of p_0.

By the construction described above, we obtain a generalized Petri net. The initial position of the game is the marking situation with only one token located at the root of the tree. The first turn is Player's if the root location is positive, while the first turn is Opponent's if the root location is negative.

Figure 4.1 shows an example of a game obtained from a logical formula.

4.3.4 Winning Condition

By the definition of a move, the game ends when all tokens are located at leaves of the game board.

To define the winning condition, we need some graph theoretical terminologies.

Definition 4.3.1. *The set of edges E of a marking situation $\langle M, \sigma, \circlearrowleft \rangle$ is defined as:*

$$E = \{\langle m, m' \rangle \in M \times M \mid m \neq m'\} / \equiv,$$

where $\equiv$ is the equivalence relation: $\langle m, m' \rangle \equiv \langle m', m \rangle$.

A coherent edge is an edge connecting coherent tokens. An incoherent edge is an edge connecting incoherent tokens. Any edge is coherent or incoherent, exclusively by the definition of coherence space.

Definition 4.3.2. *A complete mathing μ of a marking situation $\langle M, \sigma, \circlearrowleft \rangle$ is a subset of the set of edges in which each token $m \in M$ finds a unique edge connecting it with another token.*

When the game ends, Player and Opponent presents a complete matching m_{P} and m_{O} on tha marking situation at the end of the game. Player wins the game the complete matching μ_{P} satisfies all the conditions below:

1. Any edge in μ connects incoherent tokens located at dual atomic formulae.
2. For any two edges $\langle m_0, m_1 \rangle$ and $\langle m'_0, m'_1 \rangle$, either m_0 or m_1 is coherent with either m'_0 or m'_1.
3. For any alternating cycle C of edges in μ and coherent edges, the tokens consisting C are split into more than one connected components when all coherent edges are removed.

Opponent's winning condition is similar. In the next subsection, we show that Player's and Opponent's winning conditions contradict.

4.3.5 Winning Condition is equivalent to Danos-Regnier Criterion for MLL Proof Nets

The winning condition of the game is defined so as to match the Danos-Regnier criteria for MLL (Multiplicative Linear Logic) proof nets. We define a proof net as a proof structure satisfying the Danos-Regnier criterion, and show that the winning condition is equivalent to the criterion.

Definition 4.3.3 (Proof Structure and Switch)**.** *A proof structure of a play is a graph obtained by adding a complete matching μ (satisfying the conditon 1. of the winning condition) to the* *evolution structure of a play of a Linear Petri Game. A switch of a proof structure can be obtained by removing one of the two events flowing out from each token located at a negative transition and the following glue location and its exit arc (if such a glue location exists).*

If a proof structure does not exist, neither Player nor Opponent wins the game. On a proof structure, if there are n tokens located at a negative transition, there are 2^n switches.

Definition 4.3.4 (Danos-Regnier criterion)**.** *A proof structure is a proof net when all switches are connected and acyclic.*

A proof structure represents a proof of an MLL formula if and only if it is a proof net [7]. We state that the winning condition is equivalent to the Danos-Regnier criterion.

Proposition 4.3.5. *For two edges $\langle m_0, m_1 \rangle, \langle m'_0, m'_1 \rangle$ in μ satisfying the winning conditions, there are either one or four coherent edges between the two.*

Proof. By winning condition 2, the number of cohrent edges between the two matching edges is not zero. If there are two coherent edges, they either form a cycle contradicting the condition 3, or they are an impossible outcome for a tree-like evolution structure. If there are three coherent edges, they form an impossible outcome for a tree-like evolution structure. Note that any token-wise restriction of a possible outcome for a tree-like evolution structure can be split into more than one connected components either by removing all coherent edges or by removing all incoherent edges if they contain more than one tokens. ◻

Lemma 4.3.6. *The winning conditions imply that the evolution structure of the play is a proof net.*

Proof. (connectedness) If a switch of the evolution structure of the play is not connected, there is a latest removed event e (latest in the proceeding of the game, possibly not unique) that connects different connected components of the switch. There is a descendant d of the token produced in e. The other event e' flowing out from the same negative transition yield a descendant leaf token f that is not connected to d. By winning condition 2, there are coherent tokens d' and f' in the same μ-edge as d and f respectively. The coherence between d' and f' yields contradiction.

(acyclicity) If a cycle exists in a switch, the leaf tokens contained in the cycle form a counter example for winning condition 3. ◻

Lemma 4.3.7. *If a proof structure of a play is a proof net, the play satisfies the winning conditions.*

Proof. (1.) By the existence of a proof structure.

(2.) If 2. does not hold, there is a switch that does not connect the two edges, contradicting the connectedness condition.

(3.) If 3. does not hold, the counter example for 3. represents a cycle C in the evolution structure. The tokens contained in the cycle C are

1. located at $\otimes$ and unbreakable,
2. located at $⅋$ but C does not contain both branching out events, or
3. located at a leaf and connected with another dual leaf token by a μ-edge.

Since any combination of $⅋$-choice appears as a switch, there is a switch that contains C. □

Theorem 4.3.8. *The following two conditions are equivalent for a play of the game:*

1. *Player wins the game in the play, and*
2. *The evolution structure of the play is a proof net.*

Proof. By Lemma 4.3.6 and Lemma 4.3.7. □

By Theorem 4.3.8, we can conclude that both players never win the game at the same time. For any MLL formula F, since MLL proof net is sound, there are no proofs both for F and for $F^{\perp}$. If there is a play of the game $[\![F]\!]$ that satisfies both Player's and Opponent's winning conditions, the play can be interpreted both as a proof net of F as well as a proof net of $F^{\perp}$. This never happens.

4.4 Proof is Winning Strategy

The logic MLL is a sublogic of MALL. When compared with MALL, MLL simply lacks the additive connectives $\& and \oplus$, and deduction rules named after these additive connectives.

A proof π of the formula F, if read from the bottom to the leaves, describes a winning strategy on the game $[\![F]\!]$. Informally, a sequent in a proof corresponds to a possible combination of game position and strategic decision.

- A sequent of the form $\vdash \Gamma \Downarrow F$ denotes a game position where
  1. $\Gamma \cup \{F\}$ forms a $\asymp$-clique
  2. It is Player's turn
  3. Player has decided to move F.

- A sequent with $\vdash \Gamma \Uparrow \Theta$ denotes a game position where

    1. $\Gamma \cup \Theta$ forms a $\asymp$-clique
    2. It is Player's turn if and only if Θ is empty and there is a synchronous formula in Γ.

When a game is played following a focusing proof, the play becomes a restriction of the focusing proof. All ocurrances of $\oplus$ rules and & rules are removed. So, the play represents a proof of MLL and the evolution structure is a proof net when the complete matching μ is defined as the ocurrances of (I) rules.

Theorem 4.4.1. *A proof π for the formula F defines a Player's winning strategy $[\![\pi]\!]$ on the game $[\![F]\!]$.*

Proof. The strategy chooses an arc flowing out from a positive place according to $\oplus$-rules in the proof. This defines Player's evolutions. The complete matching at the end of the game is chosen as the (I)-rule occurrences in the proof. The evolution structure can be obtained by cutting off one branch from each $(\&)$-rules of the proof π. A MLL proof net can be obtained by replacing the formulae containing $\oplus$ or & with subformulae corresponding to the remaining branches. Since a MLL proof is obtained, the play is Player's win. $\square$

Corollary 4.4.2. *If there is a winning strategy for the game $[\![F^\perp]\!]$, that is the interpretation of the dual of the formula F, the formula F is unprovable.*

Proof. By Theorem 4.4.1, if there is a proof π for F, there is a winning strategy $[\![\pi]\!]$ for Player in the game $[\![F]\!]$. Since the difference between the two games $[\![F]\!]$ and $[\![F^\perp]\!]$ is just swapping of Player and Opponent, this is impossible. $\square$

4.5 Discussion

This game is very similar to the game given in [5] although there is a difference: in the game presented here, every game position is either Player's or Opponent's turn, while the game in [5] permits some game position where both Player and Opponent can make a move. Although the interpretation given in [5] interpretes a MALL formula as a function that combines games associated with atomic formulae and produces another game, the game presented in this chapter interpretes a MALL formula as a single game.

The game model presented here has not been extended to wider classes of general Petri nets. The awkwardness of the integrity condition and the winning condition suggests trying other possible choices of primitive locations, such as the one which behaves like a positive transition for tokens entering but behaves like a negative place for tokens leaving. The connection of this game semantics and the coherence space semantics of linear logic are to be investigated.

Chapter 5

Possible Applications

5.1 System-Program Interaction

The linear Petri game can be interpreted as a game between an operating system and user processes. Player is the system and Opponent is the user processes.

Positive Transition and Negative Place The `fork(2)` implementation of Linux 2.6 does `copy_process()`. This can be depicted as moving a token out of a positive transition, cloning it. The coherence between the two tokens represents the concurrent situation in which the parent process and the child process coexist. On the other side, the `fork(2)` caller gets nondeterministic return value. This passive nondeterminism is represented by a negative place, the dual of the positive transition.

Negative Transition and Positive Place A user process sometimes makes unpredictable system calls with unpredictable arguments such as when the system call is a result of user's input. This can be depicted as a token moving out of a negative transition because such a token choses an arc unpredictably (following Opponent's will). On the other side, the user process can choose which system call with what arguments. This active nondeterminism is represented by a positive place.

5.2 Visual Representation for Quantum System Evolution

In this chapter, we show that the concept of generalized Petri net models some aspects of quantum systems, both its states and dynamics. We present examples containing phenomena peculiar to quentum systems, such as creation of entanglements and measurements.

We introduce coherence relation between subspaces of an inner product space, treating orthogonal subspaces as strictly incoherent. Since a unitary transformation preserves the inner product, orthogonal subspaces remain orthogonal after a unitary transformation. The evolution of marking situations on a generalized Petri net captures this preservation of orthogonality.

The algebraic and logical treatment of Hilbert subspaces can be traced back to quantum logic presented by Birkhoff and von Neumann. A detailed description of the quantum mechanics in this line can be found in [8]. They focus their attention on the distinguishability of different static quantum states. We present a possible approach of treating dynamics of quantum systems symbolically.

5.2.1 Representation of State

In this chapter, we only treat pure states, the states whose density matrix forms a projection operator[1] P of a fixed Hilbert space $\mathbb{H}$. The class of such states are closed under unitary transformations. The projection operator P can be characterized by its image subspace $L = P(\mathbb{H})$. When L is one-dimensional, the state is conventionally represented by a unit vector $|\phi\rangle$ contained in L, so we represent any considered state by the subspace L. Actually, all L in this section can be turned into $|\phi\rangle$ safely, but losing some generality. Since the model we give in this chapter is discrete, it does not matter whether the subspace L is closed under limit. We denote the orthogonality relation between subspaces as $L \perp L'$.

To model a quantum system, we have a special kind of diffusive net, a generalized Petri net whose transitions are positive and whose places are negative.

Definition 5.2.1 (Quantum Net)**.** *A quantum net is a diffusive net whose place does not have more than one arcs flowing into it.*

Definition 5.2.2 (Interpretation for Quantum Net)**.** *An interpretation I of a quantum net is a function that maps places of the net to subspaces of the Hilbert space $\mathbb{H}$.*

Since $I(p)$ defines a subspace of $\mathbb{H}$, we obtain the unique projection operator onto the subspace $I(p)$, denoted $I_{\mathrm{P}}(p)$.

Now we have all concepts needed to write the condition for the static representation of the state L. A state L is represented by a marking situation $\langle M, \frown, \sigma \rangle$ and an interpretation I when all these three conditions hold:

1. all tokens in M are located at places,
2. a place $p \in P$ holding a token is interpreted as a subspace not orthogonal to L, that is, the relation $I(\sigma(m)) \not\perp L$ holds for any $m \in M$, and

[1] A projection operator P is defined as an Hermitian idempotent linear operator.

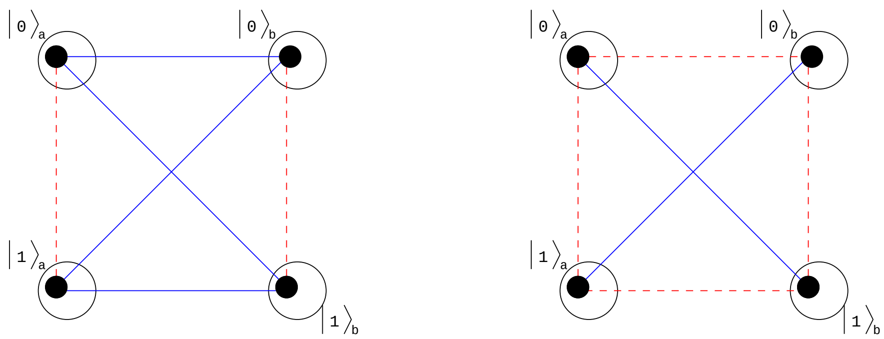

Figure 5.1: Different marking situations on the same quantum net with the same interpretation. Two qubits a and b are are not entangled (left) as in $|\phi_l\rangle = \frac{1}{2}(|0\rangle_a |0\rangle_b + |0\rangle_a |1\rangle_b + |1\rangle_a |0\rangle_b + |1\rangle_a |1\rangle_b)$. Two qubits a and b are entangled (right) as in $|\phi_r\rangle = \frac{1}{\sqrt{2}}(|0\rangle_a |1\rangle_b + |1\rangle_a |0\rangle_b)$.

3. tokens $m, m' \in M$ are coherent if and only if the projections are not orthogonal: $(I_P(\sigma(m))(L)) \not\perp (I_P(\sigma(m))(L))$.

Immediately by the form of the condition, for two marking situations with one including the other, $\langle M', \frown\restriction_{M'}, \sigma \restriction_{M'}\rangle$ represents L if $\langle M, \frown, \sigma\rangle$ represents it (the inclusion $M' \subset M$ is assumed). In other words, the presence of tokens has a meaning but the absence of tokens does not have a meaning. Especially, a marking situation containing no tokens represents any quantum state on any interpretation. Especially, a marking situation with no tokens is represents any quantum state.

Figure 5.1 shows two qubits in entanglement and not in entanglement.

5.2.2 Representation of Dynamics

In this subsection, we depict quantum system evolution as marking situation evolution. An important property of (and the motivation for defining) quantum nets is the preservation of strict incoherence.

Theorem 5.2.3 (Preservation of Strict Coherence)**.** *In an evolution on the quantum net, the children of strictly incoherent tokens are strictly incoherent.*

Proof. There are two cases: when the evolution is place-to-transition one and when the evolution is transition-to-place one. When the evolution is place-to-transition, the children of strictly incoherence tokens are strictly incoherent by Definition 3.1.5. When the evolution is transition-to-place, a produced token has only one parent because not more than one arcs flow

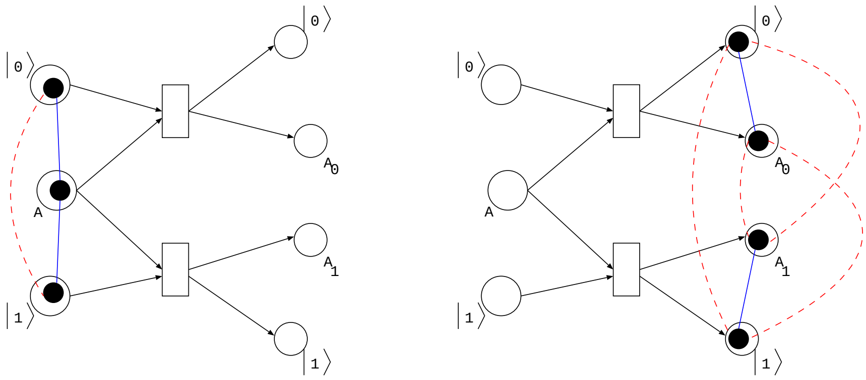

Figure 5.2: Alice and a qubit: before the measurement (left), after the measurement (right). Tokens in the places labelled A, A_0 and A_1 are Alices. Polarities are omitted because all transitions are positive and all places are negative by the definition of quantum net.

into a place on a quantum net. So, two produced tokens are coherent if and only if the parents are coherent. □

The preservation of strict coherence implies, tokens representing orthogonal subspaces remain orthogonal in an evolution sequence. This makes quantum nets suitable for depicting the quantum system evolution.

5.2.3 Examples

Qubit Observation Figure 5.2 shows a marking situation evolution depicting an observer (Alice) and a qubit before and after Alice performs a measurement on the qubit.

Observation of Entangled Pair of Qubits In Figure 5.3, Alice and Bob observe an entangled pair of qubits.

5.2.4 Discussion

It is likely that any quantum net, with an appropriate interpretation, represents a theoretically implementable structure of Hilbert space and some unitary conversions on it, we have not proved it yet.

It is not clear whether or not the duality presented in Subsection 3.2.1 has some physical meaning in this visual representation of quantum system evolution. Neither, it is not clear whether there is a natural rationale for the restriction on the number of arcs flowing into a place.

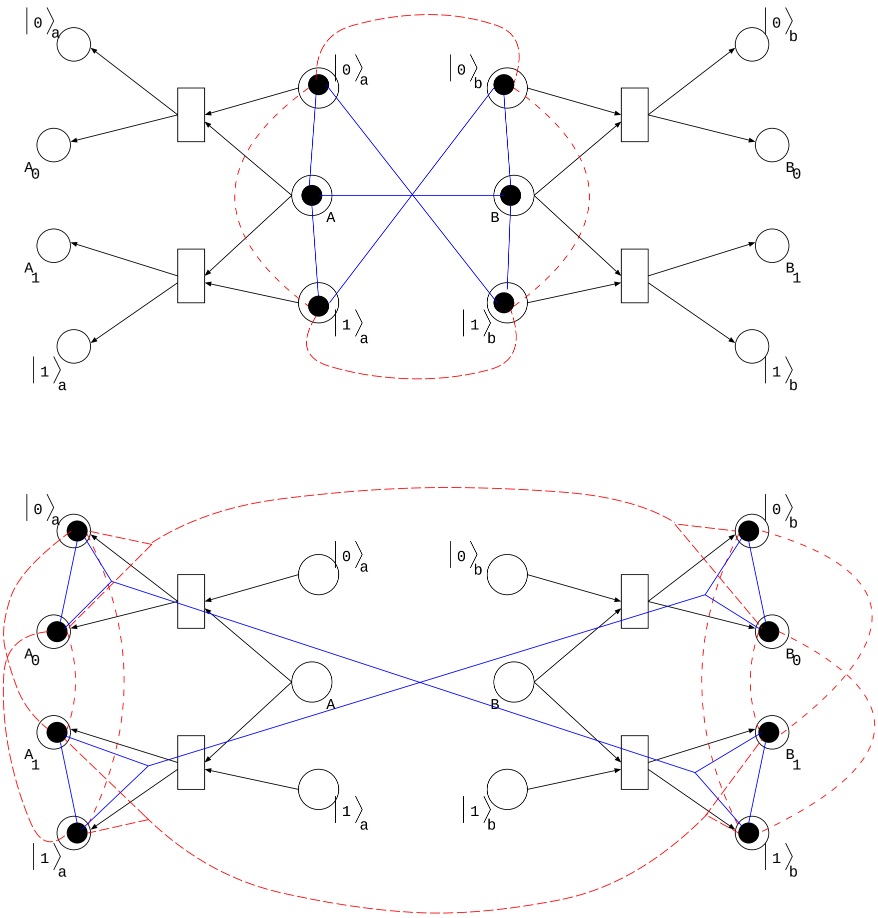

Figure 5.3: Alice, Bob and a pair of entangled qubit: before the measurement (above), after the measurement (below). The tokens in the places labelled A, A_0 and A_1 are Alices. The tokens in the places labelled B, B_0 and B_1 are Bobs. Polarities are omitted.

The visual representation presented here can be extended to general mixed states if we consider an arithmetic weighted average of marking situations because any general mixed states can be represented as an arithmetic weighted average of projection operators.

Chapter 6

Related Work

6.1 Duality for Marked Petri Nets

There is no previous work in which the duality for marked Petri nets is obtained using the notion of coherence space although many authors have invented state transition systems based on Petri net. Some transition systems allow tokens to reside on transitions of a Petri net, but the role of tokens in transitions and tokens in places are different. For example in the system of Lautenbach, K. [9], the tokens in places move along arcs when a transition fires; while the tokens in transitions move along arcs backward when a place fires. Lautenbach explains the reason for the existence of two types of tokens as:

> Supposing now, tokens on transitions (t-tokens in short) will be permitted: How and by what are they moved? What is their meaning? Moreover, a real enrichment by the t-tokens is only then conceivable if in a net both types of tokens, t-tokens and “normal” tokens (place tokens, p-tokens), are permitted. Otherwise, the dual marked net is nothing but another form of the original one [9].

The generalized Petri net presented here is a counter example to this claim because only one kind of tokens exists. Instead, the generalized Petri net has two kinds of relations between tokens: coherence and incoherence.

6.2 Linear Logic

For Petri net models, some completeness results are known for linear logic fragments ([10], [11]). However, completeness results are known only for intuitionistic fragment of linear logic, which does not incorporate MALL. For MALL, Abramsky and Melliès [5] created a game semantics, which is very similar to the Linear Petri Games described in this paper ([5]

$\otimes \quad \oplus$	$⅋ \quad \&$
Irreversible	Reversible
Active	Passive
Player	Opponent
External choice	Internal choice
Positive	Negative

Table 6.1: Curien's Summary of Linear Connectives (quoted from [13])

used Petri nets to illustrate the game model although the winning condition was not specified concretely). The game in [5] is clearly more distant from ordinary games than the Linear Petri Game in that they permit game positions in which both players can make a move. Melliès [12] recently established fully complete model of full propositional linear logic, a logic that incorporates MALL and is capable of expressing the intuitionistic propositional logic and classical propositional logic. The game in [12] is defined in an abstract manner and its properties as a model of linear logic is shown by means of categorical argument.

Curien [13] summarizes the opposition of the positive connectives ($\otimes$ and $\oplus$) and the negative connectives (& and ⅋) in Table 6.1. The Linear Petri Game in Chapter 4. depicts directly and concretely the opposition of the polarities of the linear connectives.

6.3 Logical, or Symbolic Treatment of Quantum Mechanics

The logical treatment of quantum mechanics dates back to quantum logic presented in 1936 by Birkhoff and von Neumann [14]. Quantum logic is an algebraic representation of the orthogonality and inclusion of Hilbert subspaces [8]. The algebra of quantum logic is different from Boolean algebra in that the former does not have distributivity. Linear logic also lacks distributivity for $\otimes$ and ⅋ although the two connectives are dual by negation.

V. Pratt [15] suggests using linear logic as a dynamic expansion of quantum logic. He argues that quantum logic cannot express quantum state evolution and Heisenberg uncertainty tradeoff, and that linear logic can express some aspects of state evolution and the uncertainty tradeoff by an analogy with linear automata. The relation between Pratt's interpretation of linear logic and the generalized Petri net representation of quantum systems presented here is to be investigated.

Chapter 7

Conclusion

We have defined the generalized Petri net and shown that it has a high duality wholly incorporating its dynamics as well as statics (Chapter 3). By constructing a game semantics for MALL, we found that four primitive locations of generalized Petri nets correspond to the four connectives of linear logic (Chapter 4). The duality and expressiveness of the generalized Petri net provide many applications like those presented in Chapter 5, and give deep insight into possible uses of linear logic (Chapter 6).

This work also yields awaiting investigations especially: a practical modeling of system interfaces using generalized Petri nets (or possibly using the linear type system), and the relation between generalized Petri nets and the dynamic extension of quantum logic are to be sought.

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